Commuter Arrivals and Optimal Service in Rail Transit: Does Queuing Behavior at Train Stops Matter?

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1. Introduction

Queuing behavior exhibited by passengers at urban transit stops varies widely. In the United States, passengers for the main part mill about transit stops. Earlier arriving passengers do not establish priority over passengers arriving later. We will refer to this type of behavior as random access queuing. In Japan, passengers who arrive for the bullet train establish their priority by queuing up at its stops. This is a first-in-first-out or FIFO queuing discipline. A question then arises as to whether this difference matters at all for economic policy, specifically pricing and service provision in urban mass transit. This will be the topic of the present paper.

To look at this question we employ the basic framework in Kraus and Yoshida[2], which is a model of FIFO queuing. We extend their model to the case of random access queuing and also adopt their FIFO queuing model to allow for late arrivals so that it can be compared with random access queuing. We then ask, “how are optimal pricing and service affected by the assumed queuing discipline?” We find that there are two distinct cases, depending on how the shadow value per unit of waiting time (α) compares with the shadow value per unit of time late for work (γ). Surprisingly, when α is greater than γ, the total time cost of commuters and the optimal pricing and service are the same regardless of whether the queuing is FIFO or random access. As a result, passengers’ queuing behavior in this case has
no effect on the transit authority’s policy. On the other hand, when \( \alpha \) is less than \( \gamma \), the assumed queuing discipline has an definite effect on the pattern of commuter arrivals and on the aggregate time costs that commuters incur. We prove that for any given number of commuters and service policy of transit authority, the aggregate time costs of commuters are lower under random access rather than FIFO queuing. In this case, efficient pricing and service policy will depend on the specific queue discipline that applies.

The question naturally arises as to how \( \alpha \) compares with \( \gamma \) in practice. We examine the existing empirical evidence, and find that estimates of \( \alpha \) and \( \gamma \) are very close. Thus both of the cases having to do with \( \alpha \) and \( \gamma \) are important in practice.

In section 2, we set out the general model and specialize it to the case of FIFO queuing. In section 3, we extend the model to random access queuing and analyze the two cases \( \alpha > \gamma \) and \( \alpha < \gamma \). Section 4 concludes.

2. Model with FIFO Queuing

The model described in this paper is as follows. There are \( N \) identical passengers who commute from one end of a rail line to the other, \( A \) and \( B \) respectively, between which there are no intermediate stops. \( N \) is initially taken as given, until we introduce a price-sensitive demand. The focus is on the morning commute, which moves in the direction of \( A \) to \( B \). Commuters have the common work-start time \( t^* \), and arriving earlier or later entails a schedule delay cost, with \( \beta \) and
\( \gamma \) denoting the schedule delay cost per unit of time early and late, respectively. Another cost that the commuter may incur is waiting for a train at \( A \), for which the unit time cost is \( \alpha \). We assume \( \beta < \alpha \) and \( \beta < \gamma \).

During the peak period, there are \( R \) runs (a run is a train departure from \( A \)) every \( h \) minutes. A train takes \( T \) minutes to complete a round trip where the inbound trip takes 0 minutes and the outbound (back-haul) trip takes \( T \) minutes. Having this imbalance will not be restrictive in this paper, and it is simply our way of ignoring commuter transit time. With no commuter transit time, we have the convenience of a commuter’s departure from \( A \) being her arrival time at work. Each train has a strict capacity of \( s \) passengers, so that overloading is impossible. \( K \) and \( R \) are integers, while \( s \) is continuously-valued. For simplicity, \( R \) and \( s \) are assumed to be such that the total capacity provided over the \( R \) runs, \( Rs \), is just enough to accommodate the \( N \) passengers:

\[
Rs = N. \tag{2.1}
\]

The TA sets the number of runs \( R \) to be strictly greater than the number of trains \( K \), meaning that at least one train has to make multiple runs.\(^1\) The headway is assumed to be constant, and it is thus implied by \( K \);\(^2\)

\[
h = \frac{T}{K}. \tag{2.2}
\]

\(^1\)Most real world applications are characterized by trains making multiple runs during peak period. We therefore feel that the case \( R > K \) is more interesting than the alternative \( R = K \).

\(^2\)There is no constraint on the headway if the TA sets \( K = R \).
We denote the times that these \( R \) runs depart from \( A \) by \( t_1, \ldots, t_R \) respectively, from the first to the last run. We take it that some commuters may arrive at work later than the work-start time: the last run may arrive strictly later than \( t^* \).

Commuter arrival times at origin stop are endogenous in our model, and are assumed to satisfy what is commonly called the equal-trip-price condition, where trip price is the sum of the fare and equilibrium user cost. Also known as the Wardrop user-equilibrium condition, the condition requires that the trip prices are the same at all arrival times that are used (arrival times refer to those at the origin stop), and are no less at arrival times that are not used. We are concerned with the second-best situation in which the TA sets the uniform fare.\(^3\) When this is the case, the equal-trip-price condition reduces to the condition of equal user costs, which in our model means that the sum of waiting cost and schedule delay cost must be the same for all commuters. Thus equilibrium requires that a commuter who incurs higher schedule delay cost is compensated by the lower waiting cost.

The policy variables of the TA are \( R \) (equivalently \( s \)), \( t_1, \cdots, t_R \), and \( K \). The TA’s costs are assumed to be the sum of four terms:

\[
(n_0 + n_1 s) TR + n_2 s K + n_3 s + n_4 T, \tag{2.3}
\]

where \( n_0, \ldots, n_4 \) are positive parameters. The first term is the operating cost of runs. It assumes that there are two parts to the operating cost of each run: the

\(^3\)The authors doctoral dissertation analyzes the first best with time-varying fare.
fixed part $\nu_0 T$, which is due to the cost of the driver’s time, and the variable part $\nu_1 s T$, which is due to energy costs and capital costs from operating the fleet, as well as the cost of crew other than the driver. Furthermore, the cost of operating each run is assumed proportional to run time $T$. The second term is the fleet cost: the nonoperating capital costs of cars. We assume that the cost of a car is proportional to its capacity. The third term, terminal costs, is assumed proportional to train capacity. The last term, trackage cost, is assumed proportional to the length of the line, which is proportional to $T$\textsuperscript{4}.

Total system cost is the sum of the commuters’ time costs and the TA’s costs.

So far nothing has been assumed about the queuing principle. In the rest of section 2, we will specialize in the case of first-in-first-out (FIFO) queuing. In Section 3, it is alternatively assumed to follow the random-access queuing principle.

\textbf{2.1. Equilibrium Arrival Pattern and the Optimal Scheduling with FIFO Queuing}

As mentioned earlier, in user equilibrium, the user cost for each passenger, i.e., the sum of the waiting time cost and the schedule delay cost, must be the same for all passengers. Either the first or the last passenger incurs the highest schedule delay cost, and for her there is no waiting. All those who take other runs with lower schedule delay costs incur waiting by coming to the origin stop strictly earlier.

\textsuperscript{4}Train speed is constant for both inbound and outbound trips.
than train departure times, so that everyone has the same user cost. Therefore

the equilibrium time-cost of travel is given as

\[
\max \left\{ (t^* - t_1) \beta, (t_R - t^*) \gamma \right\}. \tag{2.4}
\]

To minimize the aggregate time cost, conditional on \( R \) and \( K \), the TA simply

needs to minimize this per passenger time cost of travel:

\[
\min \left\{ \max \left\{ (t^* - t_1) \beta, (t_R - t^*) \gamma \right\} \right\}. \tag{2.5}
\]

Thus, the schedule delay cost incurred by a passenger on the first run must equal

that incurred by a passenger on the last run:

\[
(t^* - t_1) \beta = (t_R - t^*) \gamma. \tag{2.6}
\]

For those who take the intermediate runs, the difference in the schedule delay

costs to above will be compensated by the difference in waiting time.

From (2.6) and the fact that \( t_R - t_1 = (R - 1) h \), we get the aggregate time

cost under optimal\(^5\) scheduling as the following:

\[
\frac{\beta \gamma}{\beta + \gamma} (R - 1) h N. \tag{2.7}
\]

By using \( h = T/K \), the aggregate time cost is denoted as a function of \( N, K, 

\(^5\)Optimal refers to the second best under the uniform fare.
and $R$:

$$C(N, R, K) \equiv \frac{\beta \gamma}{\beta + \gamma} (R - 1) \frac{TN}{K}, \quad (2.8)$$

where $C$ is the aggregate time cost under the optimal schedule.

### 2.2. Optimal Number and Capacity of Trains and Fare with FIFO Queuing

As mentioned above, the total system cost is the sum of the commuters’ time costs and the TA’s costs. Using (2.8), total system cost under FIFO queuing are

$$\frac{\beta \gamma}{\beta + \gamma} (R - 1) \frac{TN}{K} + (\nu_0 + \nu_1 s) TR + \nu_2 sK + \nu_3 s + \nu_4 T. \quad (2.9)$$

By using $s = N/R$, (2.9) becomes a function of $R$, $K$, and $N$, which we denote by $\Phi$:

$$\Phi(N, R, K) \equiv \frac{\beta \gamma}{\beta + \gamma} (R - 1) \frac{TN}{K} + \nu_0 TR + \nu_1 TN + \frac{\nu_2 KN}{R} + \nu_3 \frac{N}{R} + \nu_4 T. \quad (2.10)$$

The remaining problem for the TA is to choose $R$ and $K$ given $N$ to minimize the total system costs:

$$\min_{R,K} \Phi(N, R, K). \quad (2.11)$$
Solving above yields the optimal values of $R$ and $K$ given $N$. Then we can define $\Gamma (N)$ as the long-run total cost given as the following:

$$
\Gamma (N) \equiv \Phi (N, R(N), K(N))
$$

(2.12)

where $R(N)$ and $K(N)$ are the optimal values given the total number of trips $N$.

The preceding analysis treats $N$ as given. The model is completed by introducing price-sensitive demand, and investigating the second-best fare that maximizes the social benefit. We let $N$ be sensitive to the equilibrium trip price, denoted by $P$,

$$
N = N(P),
$$

(2.13)

and assume that

$$
N'(P) < 0.
$$

(2.14)

Equation (2.14) implies that there exist the inverse demand function:

$$
P = P(N).
$$

(2.15)

We define social benefit as

$$
\int_0^N P(n) \, dn.
$$

(2.16)

The optimal trip price $P^*$ must be such that

$$
P^* = P(N^*)
$$

(2.17)
where

\[ N^* = \arg \max_N \int_0^N P(n) \, dn - \Gamma(N) \, . \quad (2.18) \]

The optimal fare is the difference between \( P^* \) and the equilibrium user cost at \( N^* \).

3. Model with Random-Access Queuing

We now turn our consideration to random-access queuing, again exploiting the fact that under equilibrium arrival pattern for commuters, and individual who experiences a lower schedule delay cost will have to incur a higher waiting cost. Different runs will involve different schedule delay costs, so some passenger will have to incur waiting cost in equilibrium. What form of this waiting take? Unlike the equilibrium arrival pattern under FIFO queuing, it cannot take the form of passengers arriving at \( A \) at times other than \( t_1, \ldots, t_R \). This is because all passengers who are at \( A \) when the train is about to depart have the same probability of successfully boarding the train, so that a passenger who arrives at \( A \) at a particular time establishes no train-boarding priority over a later arriving passenger.

As an illustration, suppose that a passenger is considering arriving at \( A \) at some time \( t' \) such that \( t_1 < t' < t_2 \). The earliest train that the passenger can board is the one scheduled at \( t_2 \). The passenger cannot possibly be in an equilibrium with arrival time \( t' \), because individual could switch her arrival time to \( t_2 \) without any change in likelihood of being able to board the train at \( t_2 \), while at the same time being able to eliminate the wait from \( t' \) to \( t_2 \).

Let us define \( E[c(t_i)] \) as the expected trip cost for a passenger who comes
to $A$ at $t_i$. Also, let us define $p_i$ as the probability that a passenger succeeds in getting on the $i$th train given that the passenger is waiting for the train; hence the probability of failing to board is $1 - p_i$. We assume that $p_i$ is the ratio between the train capacity $s$ and the number of passengers waiting for the $i$th train.

The expected trip cost for a passenger who tries to get the first train, denoted by $E[c(t_1)]$, is then as follows:

$$E[c(t_1)] = p_1 c^{SD}(t_1) + (1 - p_1)(h \alpha + E[c(t_2)]),$$  \hspace{1cm} (3.1)$$

where recall that $\alpha$ is the shadow price of waiting per unit time and $c^{SD}(t)$ is the schedule-delay cost incurred by a passenger who arrives at the destination at time $t$. The first term in (3.1) corresponds to the case in which a passenger succeeds in getting on the train that departs as soon as she gets to the station at $t_1$ with no waiting. In this case, the user cost is only the schedule-delay cost of arriving at work at $t_1$. This success case obtains with a probability $p_1$; in other words, there are $s/p_1$ passengers waiting for the first train. The second term is, then, the failure case: the passenger now has to wait for a time equal to the duration of headway $h$, each unit of which entails a waiting cost of $\alpha$ for her, and she then has to join the crowd waiting for the second train. Because of the absence of priority in train boarding, the additional expected trip cost for her, conditional on her being in the crowd at time $t_2$, is the same as that for those who come to the station at $t_2$ to catch the second train, and it is given by $E[c(t_2)]$.

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*If $t \leq t^*$ then $c^{SD}(t) = (t^* - t) \beta$, while if $t^* \geq t$ then $c^{SD}(t) = (t - t^*) \gamma$.**
Generalizing the above yields the expected trip cost for a passenger who comes
to the station at $t_i$ as

$$ E[c(t_i)] = p_i c^{SD}(t_i) + (1 - p_i) (h\alpha + E[c(t_{i+1})]) $$  \hspace{1cm} (3.2) 

for all $i \in \{1, \ldots, R\}$, where $E[c(t_{R+1})] = \infty$ implying that missing the last train
results in an infinite travel-time cost, because after $R$ runs, there is no means to
get the passenger to her destination. Therefore, for $i = R$, we have from (3.2) that

$$ E[c(t_R)] = p_R c^{SD}(t_R) + (1 - p_R) (h\alpha + \infty). $$  \hspace{1cm} (3.3) 

In order for the equilibrium expected trip cost to be finite, the probability of
boarding the last train should be unity: none will be bumped out of the last train.
This implies that the equilibrium expected trip cost is equal to the schedule-delay
cost incurred by arriving at work on the last train:

$$ E[c(t_R)] = c^{SD}(t_R). $$  \hspace{1cm} (3.4) 

As we stated in introduction, the implication of random-access queuing are
very different depending whether $\alpha > \gamma$ or $\gamma > \alpha$. We will now bring out why this
is so. In doing so, it will be useful to establish the following delineation of cases
for random access queuing:

Random Access 1 : $\alpha > \gamma$

Random Access 2 : $\alpha < \gamma$.

Let us start by considering random access 1, the case in which $\alpha > \gamma$. In section 3.1, we will prove, under this condition, that each run has the same expected trip cost:

$$E[c(t_1)] = \cdots = E[c(t_R)].$$

(3.5)

But in random access 2, (3.2) does not generally hold. To see this most simply, we consider a case in which the TA finds it optimal to schedule at least two late runs, i.e., $t^* < t_{R-1} < t_R$, and prove by contradiction that

$$E[c(t_{R-1})] < E[c(t_R)].$$

(3.6)

**Proof:** Suppose that $E[c(t_{R-1})] \geq E[c(t_R)]$. Then (3.2) together with (3.4) implies the following:

$$c^{SD}(t_R) \leq p_{R-1}c^{SD}(t_{R-1}) + (1 - p_{R-1}) \left(h\alpha + c^{SD}(t_R)\right).$$

(3.7)

Using the fact that $c^{SD}(t_{R-1}) = c^{SD}(t_R) - h\gamma$ rearranges the above as

$$p_{R-1} \leq \frac{\alpha}{\alpha + \gamma}.$$
Note that, in random access 2, we have $\alpha < \gamma$ and therefore the above equation implies that

$$p_{R-1} < \frac{1}{2},$$

(3.9)

which further implies that there are more than $2s$ passengers waiting at time $t_{R-1}$. But there are only $2s$ units of capacity provided from time $t_{R-1}$, so we have a contradiction to the fact that there should be no carry over after $R$ runs.

The preceding result means that (3.5) does not necessarily hold in an equilibrium arrival pattern under random access 2. This means that there going to be important differences between the nature of an equilibrium arrival pattern under random access 2 versus random access 1. When (3.6) holds, expected trip costs at $t_R$ will exceed those at $t_{R-1}$, and there will be no commuter arrivals at $t_R$. More generally, in random access 2, commuter arrivals do not necessarily occur at all train departure times. Under random access 1, on the other hand, (3.5) always holds in an equilibrium arrival pattern. This means that commuter arrivals are likely to occur at each of times of $t_1, \ldots, t_R$.

There are two distinct cases to consider, but both worth considering. The empirical evidence suggests “yes,” because it indicates that whether $\alpha > \gamma$ or $\gamma > \alpha$ is a close call. The empirical evidence on $\alpha$ is due to Quarmby[5]. Quarmby found that the value of waiting time is greater than value of in-vehicle time, \footnote{Likely, because it is theoretically possible to have equal expected user costs at two distinct times, with no arrivals occurring at one of them.}
obtaining a point estimate of ratio of the value of waiting time to the value of in-vehicle time of 2.75. He concluded from this that waiting time is worth between 2 and 3 times in-vehicle time. The empirical evidence on $\gamma$ is due to Small [6]. Small found that a late arrival at work involves a cost equivalent to 5.5 minutes of in-vehicle time, plus 2.4 minutes for each minute late. The implies cost for an individual who is 15 minute late is equivalent to 41.4 minutes of in-vehicle time. This gives a cost per minute late of $41.4/15 = 2.76$. Based on this empirical evidence, neither of the cases $\alpha > \gamma$ and $\gamma > \alpha$ can be dismissed on the grounds of being practically unimportant.

3.1. Random Access 1

We begin by noting that there must be commuter arrivals for the first run. The reason is simply that the capacity that the TA provides on the first run cannot go unused. It follows that if $\bar{c}$ denotes the expected trip cost in equilibrium, then

$$E[c(t_1)] = \bar{c}. \quad (3.10)$$

We next establish that commuter arrivals occur at all of the other train departure times $t_2, \cdots, t_R$, which means that

$$E[c(t_i)] = \bar{c} \quad (3.11)$$
for all $i \in \{1, \cdots, R\}$. Unfortunately, proving this is both lengthy and complicated, we have relegated to the appendix. Using (3.11) and (3.2) gives

$$\bar{c} = p_i c^{SD}(t_i) + (1 - p_i)(h\alpha + \bar{c}). \tag{3.12}$$

Solving (3.12) for $p_i$ and using $\bar{c} = c^{SD}(t_R)$ (from (3.11) for $i = R$ and (3.4)) gives

$$p_i = \frac{h\alpha}{c^{SD}(t_R) - c^{SD}(t_i) + h\alpha}. \tag{3.13}$$

Equation (3.13) gives us very interesting result. It tells us that how $p_i$ differs between two runs. It is determined entirely by which of the run has the higher schedule delay cost. (3.13) is an increasing function of $c^{SD}(t_R)$. This means that a run closest to $t^*$ in terms of the schedule delay cost will have the lowest probability of successful boarding, and as one moves away from $t^*$ in either direction the probability of successful boarding will monotonically increase. We include this result in the following proposition.

**Proposition 1:** In an equilibrium arrival pattern under random access 1, individuals arrive at origin stop at all times $t_1, \cdots, t_R$. Let $j \in \{1, \cdots, R\}$ be the number of run for which schedule delay cost is lowest over all runs. Then $p_1 > p_2 > \cdots > p_j$ while $p_j < p_{j+1} < \cdots < p_R$. 

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From here, we shall establish a series of results which will enable us to see the nature of an optimal schedule under random access 1. The first result is

\[ c_{SD} (t_i) \leq \bar{c} \]  \hspace{1cm} (3.14)

for \( i \in \{1, \cdots, R-1\} \). (3.14) can easily be seen by looking back (3.12) Equation (3.12) states that \( \bar{c} \) is a weighted average of \( c_{SD} (t_i) \) and \( h\alpha + c \). \( \bar{c} \) must therefore be in between the two, and therefore, no smaller than \( c_{SD} (t_i) \). This is all that (3.14) says. Next, use (3.11) for \( i = R \) and (3.4) along with (3.14) to obtain

\[ c_{SD} (t_i) \leq c_{SD} (t_R) , \]  \hspace{1cm} (3.15)

in particular,

\[ c_{SD} (t_1) \leq c_{SD} (t_R) . \]  \hspace{1cm} (3.16)

From (3.16), it is easy to see how the TA should choose the schedule. The schedule should be chosen to make \( \bar{c} \) as small as possible. From (3.11) for \( i = R, \bar{c} = E [c(t_R)] \). This means that in order for the TA to optimize the schedule, it must make \( E [c(t_R)] \) as small as possible. This is done by moving \( t_R \) leftward in the direction of \( t^* \). But in doing so, the TA is constrained by (3.16). Thus, to optimize the schedule, the TA must minimize \( c_{SD} (t_R) \) subject to (3.16). The solution to this problem is clearly

\[ c_{SD} (t_1) = c_{SD} (t_R) . \]  \hspace{1cm} (3.17)
But (3.17) is precisely the condition that we derived for optimizing schedule under FIFO queuing, where we expressed (3.17) in the form (2.6). This means that for any number of runs $R$ and headway $h$ (or, train units $K$) the aggregate time cost of commuters under optimal scheduling under random access 1 is the same as in the case of FIFO queuing and can therefore be expressed by either (2.7) or (2.8). It follows that if the queuing discipline is changing from FIFO to random access, and the case we are in is random access 1 ($\alpha > \gamma$), there is no effect on the function $\Phi(N, R, K)$ in (2.10) and therefore on $\Gamma(N)$ in (2.12). Thus there is no effect on $N^*$ (see (2.18)) and optimal fare. This gives the rather remarkable result.

**Proposition 2:** Suppose that $\alpha > \gamma$. Then whether the queuing discipline is FIFO or random access has no effect on optimal policy. The optimal number of runs, train units, scheduling, and the fare are the same in both cases.

**Remark:** It is important to realize that propositions 1 and 2 do not depend on the special structure that we imposed on the TA's costs in writing them as (2.3). Propositions 1 and 2 are quite general with respect to the TA's cost structure. The specific structure in (2.3) we specified only for expositional purpose. In a similar way the result we are about to present for random access 2 do not depend on the structure.

**3.2. Random Access 2**

As we previously indicated there will be important differences between the nature of equilibrium arrival pattern under random access 2 as compared to random
access 1. In random access 2 the latest runs generally serve only to accommodate
carry over passengers from earlier runs. The precise statement of these results is:

**Proposition 3** In an equilibrium arrival pattern under random access 2, there
is a scheduled departure time \( t_n \) such that \( E[c(t_1)] = E[c(t_2)] = \cdots =
E[c(t_n)] = \bar{c} < E[c(t_{n+1})] < \cdots < E[c(t_R)] \) and hence that individuals
arrive at origin stop at all times before \( t_n \), with no arrivals after \( t_n \).\(^8\) The
latest in the schedule that the \( n \) th run can be, is the first run after \( t^* \).

**Proof:** See appendix.

We now turn to the optimal scheduling of runs under random access 2, given
the number of passenger \( N \), runs \( R \), and train units \( K \).

**Lemma 1** Given the number of passenger \( N \), runs \( R \), and train units \( K \), the
optimal scheduling under random access 2 has the property such that

\[
E[c(t_1)] = \bar{c} = c^{SP}(t_1).
\] (3.18)

**Proof:** See appendix.

Using this lemma 1 with the fact that \( E[c(t_1)] < E[c(t_R)] = c^{SP}(t_R) \) yields
that, under the optimal scheduling,

\[
c^{SP}(t_1) < c^{SP}(t_R).
\] (3.19)

\(^8\)Except for the degenerated case, strictly positive number of passengers arrive at origin stop
at \( t_n \).
Recall that under FIFO queuing the optimal scheduling is such that $c^{SD}(t_1) = c^{SD}(t_R)$ for any number of passenger $N$, runs $R$, and train units $K$. For the same set of $R$, $K$, and $N$, under random access 2 the optimal scheduling is such that $c^{SD}(t_1) < c^{SD}(t_R)$. This further implies that $c^{SD}_{RA2}(t_1) < c^{SD}_{FIFO}(t_1)$ for the same set of $R$, $K$, and $N$, and thus $\bar{c}_{RA2} < \bar{c}_{FIFO}$.9 This gives the following:

**Proposition 4** Suppose $\gamma > \alpha$. Then for any number of passenger $N$, runs $R$, and train units $K$, the aggregate time costs of commuters under random-access queuing are less than under FIFO queuing. Thus the TA’s optimal policy will generally depend upon whether the queuing discipline is random access or FIFO.

4. Conclusions

References


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9Subscript FIFO and RA2 refer to FIFO and random access 2 queuing respectively.


A. Appendix to “Optimal Pricing and Service in Urban Mass Transit: a Contrast between Random Access and FIFO queuing” by Yuichiro Yoshida

There are three sections to this appendix. Section 1 is the appendix for Random Access 1. Section 2 is the first of two appendices for Random Access 2, and provides the proof of Proposition 3. The other appendix for Random Access 2 is Section 3, and gives the proof of Lemma 1.

A.1. Appendix to Random Access 1

In this appendix we provide the proof that commuter arrivals occur at all the train departure times \( t_2, \cdots, t_R \). We will first have to establish two lemmas.

Lemma A.1. \( p_{R-1} \geq \frac{1}{2} \).

Proof: Clearly, the probability of boarding the next to the last run, \( p_{R-1} \), must be greater than or equal to one half to have no carry over from the last run.

We will now show that \( p_{R-1} \) is strictly greater than one half.

\[
E[c(t_{R-1})] = p_{R-1}c^{SD}(t_{R-1}) + (1 - p_{R-1}) (h \alpha + E[c(t_R)])
\]

\[
= p_{R-1}c^{SD}(t_{R-1}) + (1 - p_{R-1}) (h \alpha + c^{SD}(t_R))
\]

\[
= \frac{1}{2}c^{SD}(t_{R-1}) + \frac{1}{2} (h \alpha + c^{SD}(t_R))
\]

\[
\geq \frac{1}{2} (c^{SD}(t_R) - h \gamma) + \frac{1}{2} (h \alpha + c^{SD}(t_R))
\]

\[
= c^{SD}(t_R) + \frac{1}{2} (h \alpha - h \gamma)
\]
This is feasible in equilibrium only if there is none coming to the station at $t_{R-1}$, and thus the probability of boarding the third to the last run $p_{R-2}$ must be $1/3$. However, applying this logic recursively backward shows that this cannot sustain the equilibrium. To show this we use mathematical induction.

Assume that

$$p_{R-i} = \frac{1}{i + 1}$$

and that

$$E[c(t_{R-i+1})] > E[c(t_R)]$$

for some $i \in \{2, \cdots, R - 1\}$. Then

$$E[c(t_{R-i})] = p_{R-i}c^{SD}(t_{R-i}) + (1 - p_{R-i})(h\alpha + E[c(t_{R-i+1})])$$

$$= \frac{1}{i + 1}c^{SD}(t_{R-i}) + \frac{i}{i + 1}(h\alpha + E[c(t_{R-i+1})])$$

$$> \frac{1}{i + 1}c^{SD}(t_{R-i}) + \frac{i}{i + 1}(h\alpha + c^{SD}(t_R))$$

$$\geq \frac{1}{i + 1}(c^{SD}(t_R) - ih\gamma) + \frac{i}{i + 1}(h\alpha + c^{SD}(t_R))$$

$$= c^{SD}(t_R) + \frac{i}{i + 1}(h\alpha - h\gamma)$$

$$> E[c(t_R)].$$
This is feasible in equilibrium only if there is none coming to the station at \( t_{R-i} \). Thus
\[
p_{R-i-1} = \frac{1}{i+2}.
\]

Combining this with the previous results yields

\[
E[c(t_i)] > E[c(t_R)]
\]

for all \( i \in \{1, \cdots, R - 1\} \), which cannot be an equilibrium. ■

**Lemma A.2.** \( E[c(t_i)] = \bar{c}, \) for all \( i \in \{1, \cdots, R\} \).

**Proof:** The fact that \( p_{R-1} > 1/2 \) implies that the mass of commuters waiting at \( t_{R-1} \) is strictly smaller than \( 2\bar{s} \), and therefore that there are strictly positive number of passengers arriving at origin stop at \( t_R \). This further implies that \( E[c(t_R)] = \bar{c} \). We now proceed to show that \( E[c(t_i)] = \bar{c} \) for all \( i \in \{1, \cdots, R - 1\} \). We have that

\[
E[c(t_R)] = \bar{c},
\]

and since commuters come to the origin station for the first run,

\[
E[c(t_1)] = \bar{c}.
\]
Suppose that
\[ E[c(t_i)] = E[c(t_j)] = c, \] (A.1)
for some \( i, j \in \{1, \cdots, R\} \), and that for the \( j-1 \)th run we have
\[ E[c(t_{j-1})] > c, \] (A.2)
where \( i < j-1 < j \). Define \( N_{j-1} \) as the number of passengers waiting for the \( j-1 \)th run at the origin station. Since (A.2) implies that none will come to the station at \( t_{j-1} \), the probability of boarding the \( j-2 \)th run is
\[ p_{j-2} = \frac{s}{N_{j-1} + s}. \]
Using \( p_{j-1} = s/N_{j-1} \) in above yields
\[ p_{j-2} = \frac{p_{j-1}}{1 + p_{j-1}}. \] (A.3)
Solving
\[ E[c(t_{j-1})] = p_{j-1} c^{SD}(t_{j-1}) + (1 - p_{j-1}) (h\alpha + E[c(t_j)]) \]
for \( p_{j-1} \) yields
\[ p_{j-1} = \frac{h\alpha + E[c(t_j)] - E[c(t_{j-1})]}{h\alpha + E[c(t_j)] - c^{SD}(t_{j-1})}. \] (A.4)
Using (A.1) and (A.2) in (A.3) yields

\[
\begin{align*}
   p_{j-2} &= \frac{h\alpha + E[c(t_j)] - E[c(t_{j-1})]}{h\alpha + E[c(t_j)] - c^{SD}(t_{j-1}) + h\alpha + E[c(t_j)] - E[c(t_{j-1})]} \\
   &< \frac{h\alpha + E[c(t_j)] - c^{SD}(t_{j-1}) + h\alpha + E[c(t_j)] - E[c(t_{j-1})]}{h\alpha + \bar{c} - \bar{c}} \\
   &= \frac{h\alpha + \bar{c} - \bar{c} - c^{SD}(t_{j-1}) + h\alpha + \bar{c} - \bar{c}}{h\alpha} \\
   &= \frac{h\alpha}{h\alpha + \bar{c} - c^{SD}(t_{j-1}) + h\alpha}.
\end{align*}
\]  

(A.5)

Similarly to (A.4),

\[
    p_{j-2} = \frac{h\alpha + E[c(t_{j-1})] - E[c(t_{j-2})]}{h\alpha + E[c(t_{j-1})] - c^{SD}(t_{j-2})},
\]  

(A.6)

and thus

\[
\begin{align*}
    p_{j-2}\big|_{E[c(t_{j-1})]=E[c(t_{j-2})]=\bar{c}} &= \frac{h\alpha}{h\alpha + \bar{c} - c^{SD}(t_{j-2})} \\
    &\geq \frac{h\alpha}{h\alpha + \bar{c} - c^{SD}(t_{j-1}) + h\gamma} \\
    &> \frac{h\alpha}{h\alpha + \bar{c} - c^{SD}(t_{j-1}) + h\alpha}.
\end{align*}
\]  

(A.7)

From (A.6),

\[
    p_{j-2}\big|_{E[c(t_{j-2})]=\bar{c}} \geq p_{j-2}\big|_{E[c(t_{j-1})]=E[c(t_{j-2})]=\bar{c}}.
\]  

(A.9)

Combining (A.5), (A.8), and (A.9) yields

\[
    p_{j-2}\big|_{E[c(t_{j-2})]=\bar{c}} > p_{j-2},
\]  

(A.10)
which implies

\[ E[c(t_{j-2})] > \bar{c}. \]

Starting from the above result obtained for \((j - 2)\)th run, we use induction to show that above holds for all the earlier runs. Assume that, for all \(k \in \{k, \ldots, j-2\}\) where \(k \geq i + 1\),

\[ E[c(t_k)] > \bar{c}, \quad (A.11) \]

and that

\[ p_k < p_k|E[c(t_k)] = \ldots = E[c(t_{j-2})] = \bar{c} \quad (A.12) \]

Then since none comes to the station at \(t_k\), the probability of boarding the \(k - 1\)th run is

\[ p_{k-1} = \frac{s}{N_k + s} \]

where \(N_k\) as the number of passengers waiting for the \(j - 1\)th run at the origin station. Using \(p_k = s/N_k\) in above yields

\[ p_{k-1} = \frac{P_k}{1 + P_k} < \frac{p_k|E[c(t_k)] = \ldots = E[c(t_{j-2})] = \bar{c}}{1 + p_k|E[c(t_k)] = \ldots = E[c(t_{j-2})] = \bar{c}}, \quad (A.13) \]
where the inequality follows from (A.12). Since
\[ P_k = \frac{h\alpha + E[c(t_{k+1})] - E[c(t_k)]}{h\alpha + E[c(t_{k+1})] - c^{SD}(t_k)}, \]
\[ p_k|_{E[c(t_k)]=\cdots=E[c(t_{j-2})]=\bar{\varepsilon}} = \frac{h\alpha}{h\alpha + \bar{c} - c^{SD}(t_k)}. \] (A.14)

Using (A.14) in (A.13) gives
\[ p_{k-1} < \frac{p_k|_{E[c(t_k)]=\cdots=E[c(t_{j-2})]=\bar{\varepsilon}}}{1 + p_k|_{E[c(t_k)]=\cdots=E[c(t_{j-2})]=\bar{\varepsilon}}} \]
\[ = \frac{h\alpha}{h\alpha + \bar{c} - c^{SD}(t_k) + h\alpha}. \] (A.15)

At the same time
\[ p_{k-1} = \frac{h\alpha + E[c(t_k)] - E[c(t_{k-1})]}{h\alpha + E[c(t_k)] - c^{SD}(t_{k-1})}, \] (A.16)

and thus
\[ p_{k-1}|_{E[c(t_{k-1})]=\cdots=E[c(t_{j-2})]=\bar{\varepsilon}} = \frac{h\alpha}{h\alpha + \bar{c} - c^{SD}(t_{k-1})} \]
\[ \geq \frac{h\alpha}{h\alpha + \bar{c} - c^{SD}(t_k) + h\gamma} \]
\[ > \frac{h\alpha}{h\alpha + \bar{c} - c^{SD}(t_k) + h\alpha}. \] (A.17)

Combining (A.15) and (A.17) yields
\[ p_{k-1}|_{E[c(t_{k-1})]=\cdots=E[c(t_{j-2})]=\bar{\varepsilon}} > p_{k-1}. \] (A.18)
Together with the fact that $E[c(t_k)] > \bar{c}$, (A.16) also implies that

\[ p_{k-1} \mid E[c(t_{k-1})] = \bar{c} > p_{k-1} \mid E[c(t_{k-1})] = \cdots = E[c(t_{j-2})] = \bar{c} \]  

(A.19)

Combining (A.18) and (A.19) yields

\[ p_{k-1} \mid E[c(t_{k-1})] = \bar{c} > p_{k-1}, \]

which implies

\[ E[c(t_{k-1})] > \bar{c}. \]

When $k = i + 1$,

\[ E[c(t_i)] > \bar{c}, \]

which is a contradiction. ■

We are now ready to prove the result stated both in the text and in the beginning of this appendix, namely commuter arrivals occur at all the train departure times $t_2, \cdots, t_R$.

*Prove that there are positive number of arrivals at all times $t_2, \cdots, t_R$: we already know that there are positive number of passengers arriving at $t_R$. This transforms the statement to be proved to that commuter arrivals occur at $t_2, \cdots, t_{R-1}$. We prove this by contradiction. Suppose that at time $t_i$ for
some \( i \in \{2, \cdots, R - 1\} \) there are no passenger arriving at A. Then we have

\[
p_{i-1} = \frac{p_i}{1 + p_i} \tag{A.20}
\]

and equivalently

\[
1 - p_{i-1} = \frac{1}{1 + p_i}. \tag{A.21}
\]

Using (3.11) for \( i \) and \( i - 1 \) gives

\[
E[c(t_i)] = p_i c^{SD}(t_i) + (1 - p_i) (h_\alpha + E[c(t_{i+1})])
\]

\[
E[c(t_{i-1})] = p_{i-1} c^{SD}(t_{i-1}) + (1 - p_{i-1}) (h_\alpha + E[c(t_i)])
\]

Recall that \( E[c(t_i)] = \bar{c} \) for all \( i \in \{1, \cdots, R\} \) to obtain

\[
\bar{c} = p_i c^{SD}(t_i) + (1 - p_i) (h_\alpha + \bar{c}) \tag{A.22}
\]

\[
\bar{c} = p_{i-1} c^{SD}(t_{i-1}) + (1 - p_{i-1}) (h_\alpha + \bar{c}) \tag{A.23}
\]

Applying (A.20) and (A.21) rewrites (A.23) as

\[
\bar{c} = \frac{p_i}{1 + p_i} c^{SD}(t_{i-1}) + \frac{1}{1 + p_i} (h_\alpha + \bar{c}). \tag{A.24}
\]

Multiplying the both sides of (A.24) by \( 1 + p_i \) and subtracting it from (A.22) yields

\[
c^{SD}(t_i) - c^{SD}(t_{i-1}) = h_\alpha,
\]

30
which is impossible in the case of random access 1 where \( \alpha > \gamma \). Thus we obtain a contradiction. \( \blacksquare \)
A.2. Appendix to Random Access 2

This appendix establishes two lemmas describing the equilibrium trip costs and passenger arrivals under the random access 2, one for late runs and another for early runs.

**Lemma A.3.** Define the \( l \)th run be the earliest run that arrives no earlier than \( t^* \).\(^{10}\)

\[
    l = \arg \min_i t_i \geq t^*.
\]  \hspace{1cm} (A.25)

Then for any \( i \in \{l, \ldots, R - 1\}, \)

\[
    E[c(t_i)] < E[c(t_{i+1})],
\]  \hspace{1cm} (A.26)

therefore none comes to the origin station after \( t_i \), and the probabilities of boarding are given as

\[
    p_i = \frac{1}{R - i + 1},
\]  \hspace{1cm} (A.27)

which is an increasing function with respect to \( i \).

**Proof:** We prove this lemma by using the mathematical induction in the backward direction from the last run. In the equilibrium, everyone should be

\(^{10}\)Therefore, \( l = k \) if \( t_l = t^* = t_k \), and \( l = k + 1 \) if \( t^* < t_l \) or \( t_k < t^* \), where \( k \) is the number of runs that arrive no later than \( t^* \) as defined as earlier. Also, since the \( l \)th run is the first non-early run, \( l \) is the smallest integer that satisfies the following:

\[
    l \geq 1 + \frac{c^{SD}(t_1)}{\beta}.
\]
able to get to their destination by the end of the peak period.\textsuperscript{11} This implies that there cannot be more than twice as many passengers as the train capacity, i.e., $2s$ passengers, waiting for the second to the last run. Thus the probability of boarding on the second to last run has to be greater or equal to one half:

$$p_{R-1} \geq \frac{1}{2}. \quad (A.28)$$

Similar reasoning can be generalized to obtain that there cannot be more than $js$ people waiting for the $j$th to the last train run in order to carry all commuters until the last train. This implies, in a similar manner as before, that the probability of getting on board the $j$th to the last run has to be greater than or equal to $1/j$:

$$p_{R-j+1} \geq \frac{1}{j} \quad (A.29)$$

or

$$p_{R-i} \geq \frac{1}{i+1} \quad (A.30)$$

where $i \equiv j-1$. Given these probabilities, we now look into the expected trip cost of each run in the equilibrium in order starting from the last run, i.e., the $R$th run.\textsuperscript{12} Since none will be bumped out of the last train, the expected

\textsuperscript{11}Provided that we have finite expected trip price.

\textsuperscript{12}Here we assume that there is a sufficient number of late runs to illustrate the following.
trip cost of the last run is just the schedule delay cost that it entails:

\[ E[c(t_R)] = c^{SD}(t_R). \]  \hspace{1cm} \text{(A.31)}

For the \((R - 1)\)st run, there is a possibility of carry over, and therefore the probability of getting on board the run is not necessarily unity. Therefore the expected trip cost for the \((R - 1)\)st run is

\[ E[c(t_{R-1})] = p_{R-1}c^{SD}(t_{R-1}) + (1 - p_{R-1}) \left(h\alpha + E[c(t_R)]\right) \]  \hspace{1cm} \text{(A.32)}

for some \(p_{R-1} \in [1/2, 1]\). Using (A.31) and the fact that

\[ c^{SD}(t_{R-1}) = c^{SD}(t_R) - h\gamma \]  \hspace{1cm} \text{(A.33)}

solves the expected trip cost for the \((R - 1)\)th run as the following:

\[ E[c(t_{R-1})] = c^{SD}(t_R) - p_{R-1}h(\alpha + \gamma) + h\alpha. \]  \hspace{1cm} \text{(A.34)}

Given that \(\gamma\) is greater than \(\alpha\), and that the probability of getting the \((R - 1)\)th run is greater than or equal to one half, (A.34) yields that

\[ E[c(t_{R-1})] < E[c(t_R)]. \]  \hspace{1cm} \text{(A.35)}
That is, the expected trip cost of the last run is always higher than that of the second to last run: the last run is always dominated by its immediate predecessor in terms of the expected trip cost, and therefore none leaves home just before \( t_R \) to catch the last train. In such a case, there are 2\( s \) people waiting for the \((R - 1)th\) run so the probability of getting on the \((R - 1)th\) train is actually equal to one half:

\[
p_{R-1} = \frac{1}{2}. \tag{A.36}
\]

This further implies that the expected trip cost of the \((R - 1)th\) run can now be written as

\[
E [c(t_{R-1})] = c_{SD}^R (t_R) - \frac{1}{2} h (\gamma - \alpha). \tag{A.37}
\]

We have already shown that the last run entails higher expected trip cost than does its immediate predecessor, i.e.,

\[
E [c(t_{R-1})] < E [c(t_R)], \tag{A.38}
\]

which implies that there is a maximum number of commuters waiting for the \((R - 1)th\) train:

\[
p_{R-1} = \frac{1}{2}. \tag{A.39}
\]
Thus, we see that the lemma holds for $i = R - 1$. Now we assume that $i = k \in \{l + 1, \ldots, R - 1\}$ and that for such $i$ the lemma holds as well. Then we have the following to work with:

$$E [c(t_k)] < E [c(t_{k+1})]$$  \hspace{1cm} (A.40)

and hence

$$p_k = \frac{1}{R - k + 1}.$$  \hspace{1cm} (A.41)

and we have the following to show:

$$E [c(t_{k-1})] < E [c(t_k)]$$  \hspace{1cm} (A.42)

and hence

$$p_{k-1} = \frac{1}{R - k + 2}.$$  \hspace{1cm} (A.43)

Here we know that the expected trip costs consist of two parts, namely, costs incurred when a passenger has successfully boarded and when she has failed, with each assigned the corresponding probability:

$$E [c(t_k)] = p_k c^{SD} (t_k) + (1 - p_k) (h \alpha + E [c(t_{k+1})])$$  \hspace{1cm} (A.44)

$$E [c(t_{k-1})] = p_{k-1} c^{SD} (t_{k-1}) + (1 - p_{k-1}) (h \alpha + E [c(t_k)])$$  \hspace{1cm} (A.45)
Therefore, showing (A.42) is equivalent to showing

\[
p_k \left[ c^{SD}(t_k) - E[c(t_{k+1})] - h\alpha \right] + E[c(t_{k+1})] > p_{k-1} \left[ c^{SD}(t_k) - E[c(t_k)] - h(\alpha + \gamma) \right] + E[c(t_k)]. \tag{A.46}
\]

In showing this, we look at the right-hand side of the inequality, and view it as a function of \( p_{k-1} \). It can be shown that this is decreasing with respect to \( p_{k-1} \), since by using the assumption that \( E[c(t_k)] \) is smaller than \( E[c(t_{k+1})] \), we get the following inequality:

\[
p_k \left[ c^{SD}(t_k) - E[c(t_k)] - h\alpha \right] + h\alpha < 0, \tag{A.47}
\]

which implies that

\[
c^{SD}(t_k) - E[c(t_k)] - h(\alpha + \gamma) < 0. \tag{A.48}
\]

In addition to the fact that the right-hand side is decreasing in \( p_{k-1} \), since there can be no left-over passengers after the peak period ends, \( p_{k-1} \) can take a value \( 1/(R-k+2) \) at its least. Thus, it is sufficient to show (A.42) holds when \( p_{k-1} \) is equal to \( 1/(R-k+2) \). The assumption that \( p_k = 1/(R-k+1) \) together with this yields the following inequality that we have
to show instead:

$$
\frac{1}{R - k + 1} \left[ c^{SD} (t_k) - E[c(t_{k+1})] + h\alpha \right] + E[c(t_{k+1})]
\geq \frac{1}{R - k + 2} \left[ c^{SD} (t_k) - E[c(t_k)] - h(\alpha + \gamma) \right] + E[c(t_k)] \quad (A.49)
$$
or equivalently,

\[
\frac{1}{R - k + 1} \left[ c^{SD} \left( t_k \right) - E \left[ c \left( t_{k+1} \right) \right] - h\alpha \right] \\
+ (R - k + 1) \left[ E \left[ c \left( t_{k+1} \right) \right] - E \left[ c \left( t_k \right) \right] \right] + h\gamma \tag{A.51}
\]

\[
> 0. \tag{A.52}
\]

\[
\frac{1}{R - n + 1} \left[ c^{SD} \left( t_n \right) - E \left[ c \left( t_{n+1} \right) \right] - h\alpha \right] + E \left[ c \left( t_{n+1} \right) \right] \\
> \frac{1}{R - n + 2} \left[ c^{SD} \left( t_n \right) - E \left[ c \left( t_n \right) \right] - h \left( \alpha + \gamma \right) \right] + E \left[ c \left( t_n \right) \right] \\
(R - n + 2) \left[ c^{SD} \left( t_n \right) - E \left[ c \left( t_{n+1} \right) \right] - h\alpha \right] + (R - n + 1) \left( R - n + 2 \right) E \left[ c \left( t_{n+1} \right) \right] \\
> (R - n + 1) \left[ c^{SD} \left( t_n \right) - E \left[ c \left( t_n \right) \right] - h \left( \alpha + \gamma \right) \right] + (R - n + 1) \left( R - n + 2 \right) E \left[ c \left( t_n \right) \right] \\
(R - n + 2) \left[ c^{SD} \left( t_n \right) - E \left[ c \left( t_{n+1} \right) \right] - h\alpha \right] \\
+ (R - n + 1) \left( R - n + 2 \right) E \left[ c \left( t_{n+1} \right) \right] \\
> (R - n + 1) \left[ c^{SD} \left( t_n \right) - E \left[ c \left( t_n \right) \right] - h\alpha \right] \\
- (R - n + 1) E \left[ c \left( t_{n+1} \right) \right] - (R - n + 1) \left( R - n + 2 \right) E \left[ c \left( t_n \right) \right] \\
\frac{1}{R - n + 2} \left[ c^{SD} \left( t_n \right) - E \left[ c \left( t_{n+1} \right) \right] - h\alpha \right] \\
- (R - n + 1) \left[ c^{SD} \left( t_n \right) - E \left[ c \left( t_n \right) \right] - h\alpha \right] \\
> (R - n + 1) h\gamma + (R - n + 1) \left( R - n + 2 \right) E \left[ c \left( t_n \right) \right] \\
\frac{1}{R - n + 1} \left[ c^{SD} \left( t_n \right) - E \left[ c \left( t_{n+1} \right) \right] - h\alpha \right] \\
+ (R - n + 1) \left[ E \left[ c \left( t_{n+1} \right) \right] - E \left[ c \left( t_n \right) \right] \right] + h\gamma \\
> 0. \tag{A.50}
\]
We use the following information, which we get from taking the difference of \( E[c(t_{k+1})] \) and \( E[c(t_k)] \):\(^\text{14}\)

\[
\frac{1}{R - k + 1} \left[ c^{SD}(t_k) - E[c(t_{k+1})] - h\alpha \right] = E[c(t_{k+1})] - E[c(t_k)] - h\alpha \quad (A.53)
\]

to rewrite the inequality (A.49) as the following:\(^\text{15}\)

\[
(R - n + 2) [E[c(t_{k+1})] - E[c(t_k)]] + h(\gamma - \alpha) > 0. \quad (A.54)
\]

This is clearly satisfied from the assumptions that \( E[c(t_{k+1})] > E[c(t_k)] \) and that \( \gamma > \alpha \). Thus, we have shown that \( E[c(t_{k-1})] < E[c(t_k)] \), and this

\[
E[c(t_n)] = p_n c^{SD}(t_n) + (1 - p_n) (h\alpha + E[c(t_{n+1})])
\]

\[
E[c(t_n)] - E[c(t_{n+1})] = p_n c^{SD}(t_n) + (1 - p_n) h\alpha - p_n E[c(t_{n+1})]
\]

\[
= p_n c^{SD}(t_n) + h\alpha - p_n h\alpha - p_n E[c(t_{n+1})]
\]

\[
= p_n \left[ c^{SD}(t_n) - h\alpha - E[c(t_{n+1})] \right] + h\alpha
\]

Since \( p_n = 1/(R - n + 1) \),

\[
E[c(t_n)] - E[c(t_{n+1})] - h\alpha = \frac{1}{R - n + 1} \left[ c^{SD}(t_n) - E[c(t_{n+1})] - h\alpha \right].
\]

\[\text{14}\]

\[\text{15}\]
further implies that none is willing to leave home at \( t_k \) to get the \( k \)th run, and there are maximum possible number of commuters waiting for the run at \( t_{k-1} \). Therefore, the probability of getting on the \((k - 1)\)st run is indeed at its minimum:

\[
p_{k-1} = \frac{1}{R - k + 2}.
\]

We now discuss the expected trip costs and passenger arrivals of early runs. Let us start by defining \( D(t) \) as the cumulative departure function and then finding the expected trip cost for the \( l - 1 \)th run. Since \( t_{l-1} \) is before \( t^* \), the lemma does not necessarily hold; the expected trip cost at \( t_{l-1} \) is not necessarily smaller than that at \( t_l \), and they may well be the same, i.e., \( E[c(t_{l-1})] = E[c(t_l)] \). If they are the same, the value for these expected trip costs is then the equilibrium expected trip cost, and all the earlier runs incur the same expected cost. However, even if there are a maximum number of people waiting for the \( l - 1 \)th run and hence \( D(t_{l-1}) = N \), it is possible that the expected trip cost at \( t_{l-1} \) is still smaller than that at \( t_l \). If this is the case, we have to move to the earlier runs and compare their expected trip costs: \( E[c(t_{l-1})] \) and \( E[c(t_{l-2})] \). The following proposition summarizes this point.

**Lemma A.4.** Let \( n \in \{2, \ldots, l\} \) be the largest integer for which

\[
E[c(t_{n-1})] = E[c(t_n)]
\]  
(A.55)
Then

(i) The expected trip cost for all runs earlier than the \( n \)th run is equal to that of the \( n \)th run:

\[
E [c(t_1)] = \cdots = E [c(t_n)] = \bar{c}
\]  
(A.56)

and there is a strictly positive number of passengers coming to the station for \( t_1 \) through \( t_{n-1} \).

(ii) The expected trip cost for all early runs later than the \( n \)th run is greater than that of its immediate predecessor:

\[
E [c(t_{i-1})] < E [c(t_i)]
\]  
(A.57)

for \( i \in \{n + 1, \cdots, l\} \).

**Proof:** We prove (i) by mathematical induction. We already have from the assumption that

\[
E [c(t_{n-1})] = E [c(t_n)] .
\]  
(A.58)

Assume for some \( i \in \{2, \cdots, n\} \) that

\[
E [c(t_{i-1})] = E [c(t_i)] .
\]  
(A.59)
Then we need to show that

\[ E[c(t_{i-2})] = E[c(t_{i-1})]. \quad (A.60) \]

First, suppose that

\[ E[c(t_{i-2})] < E[c(t_{i-1})]. \quad (A.61) \]

Then since none comes to the origin station at \( t_{i-1}, \)

\[ p_{i-2} = \frac{p_{i-1}}{1 + p_{i-1}}, \quad (A.62) \]

which implies that

\[ p_{i-2} < p_{i-1}. \quad (A.63) \]

Since both the \( i - 1 \)th and \( i - 2 \)th runs are early runs,

\[ c^{SD}(t_{i-1}) < c^{SD}(t_{i-2}). \quad (A.64) \]

Using (A.63) and (A.64) in the following expressions

\[ E[c(t_{i-1})] = p_{i-1}c^{SD}(t_{i-1}) + (1 - p_{i-1}) (h\alpha + E[c(t_i)]) \quad (A.65) \]

\[ E[c(t_{i-2})] = p_{i-2}c^{SD}(t_{i-2}) + (1 - p_{i-2}) (h\alpha + E[c(t_{i-1})]) \quad (A.66) \]

\[ = p_{i-2}c^{SD}(t_{i-2}) + (1 - p_{i-2}) (h\alpha + E[c(t_i)]), \quad (A.67) \]
implies that

$$E[c(t_{i-2})] > E[c(t_{i-1})],$$  \hspace{1cm} (A.68)

which is a contradiction.\(^\text{16}\) Second, suppose that

$$E[c(t_{i-2})] > E[c(t_{i-1})].$$  \hspace{1cm} (A.69)

Then none comes to the station at \(t_{i-2}\), and therefore,

$$p_{i-3} = \frac{p_{i-2}}{1 + p_{i-2}},$$  \hspace{1cm} (A.70)

which implies

$$p_{i-3} < p_{i-2}.$$  \hspace{1cm} (A.71)

Since again both the \((i - 2)\)nd and \((i - 3)\)rd runs are early runs,

$$c^{SD}(t_{i-2}) < c^{SD}(t_{i-3}).$$  \hspace{1cm} (A.72)

\(^{16}\)The expression \((h\alpha + E[c(t_i)])\) is greater than \(c^{SD}(t_{i-1})\) since

$$E[c(t_{i-1})] = p_{t-1}c^{SD}(t_{i-1}) + (1 - p_{t-1})(h\alpha + E[c(t_i)])$$

and \(E[c(t_{i-1})] = E[c(t_i)]\) imply

$$E[c(t_i)] = c^{SD}(t_{i-1}) + \frac{1 - p_{i-1}}{p_{i-1}}h\alpha.$$
Substituting (A.71) and (A.72) in the following expressions where the third line uses (A.69)

\[
E[c(t_{i-2})] = p_{i-2}c^{SD}(t_{i-2}) + (1 - p_{i-2})(h\alpha + E[c(t_{i-1})]) \tag{A.73}
\]

\[
E[c(t_{i-3})] = p_{i-3}c^{SD}(t_{i-3}) + (1 - p_{i-3})(h\alpha + E[c(t_{i-2})]) \tag{A.74}
\]

\[
> p_{i-3}c^{SD}(t_{i-3}) + (1 - p_{i-3})(h\alpha + E[c(t_{i-1})]), \tag{A.75}
\]

implies that

\[
E[c(t_{i-3})] > E[c(t_{i-2})]. \tag{A.76}
\]

Applying this logic recursively eventually yields\textsuperscript{17}

\[
E[c(t_1)] > E[c(t_i)], \tag{A.77}
\]

which contradicts with the equilibrium. Combining above results gives the desired result that

\[
E[c(t_{i-2})] = E[c(t_{i-1})], \tag{A.78}
\]

which, together with

\[
E[c(t_{n-1})] = E[c(t_n)] \tag{A.79}
\]

implies

\[
E[c(t_1)] = \cdots = E[c(t_n)]. \tag{A.80}
\]

\textsuperscript{17}The derivation of this result uses mathematical induction; however, it is excluded since it is straightforward.
Since commuters must show up for the first run, we have

\[ E [c(t_1)] = \cdots = E [c(t_n)] = \bar{c}. \tag{A.81} \]

Using the above in

\[ E [c(t_i)] = p_i c^{SD}(t_i) + (1 - p_i) (h\alpha + E [c(t_{i+1})]) \]

for \( i \in \{1, \cdots, n-1\} \) yields

\[ \bar{c} = p_i c^{SD}(t_i) + (1 - p_i) (h\alpha + \bar{c}), \]

which implies

\[ p_i = \frac{h\alpha}{h\alpha + \bar{c} - c^{SD}(t_i)}. \tag{A.82} \]

The above equation (A.82) shows that \( p_i \) is decreasing over \( \{1, \cdots, n-1\} \).

Also, using (A.82) for the \((i - 1)\)st run gives

\[ p_{i-1} = \frac{h\alpha}{h\alpha + \bar{c} - c^{SD}(t_{i-1})} \]

\[ = \frac{h\alpha}{h\alpha + \bar{c} - c^{SD}(t_i) - h\beta} \]

\[ > \frac{h\alpha}{h\alpha + \bar{c} - c^{SD}(t_i) + h\alpha} \]

\[ = \frac{p_i}{1 + p_i}, \]

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implying that there is a strictly positive number of passengers coming to the station for \( t_1, \ldots, t_{n-1} \).\(^{18}\)

Next, we prove (ii). We know that for \( i \in \{ n+1, \ldots, l \} \) that

\[
E [ c(t_{i-1}) ] \neq E [ c(t_i) ]. \tag{A.83}
\]

Suppose, for any \( i \in \{ n+1, \ldots, l \} \), that

\[
E [ c(t_{i-1}) ] > E [ c(t_i) ] \tag{A.84}
\]

then none comes to the station at \( t_{i-1} \), and therefore,

\[
p_{i-2} = \frac{p_{i-1}}{1 + p_{i-1}}, \tag{A.85}
\]

which implies

\[
p_{i-2} < p_{i-1}. \tag{A.86}
\]

Since again both the \( i-1 \)th and \( i-2 \)th runs are early runs,

\[
c^{SD}(t_{i-1}) < c^{SD}(t_{i-2}). \tag{A.87}
\]

\(^{18}\)When \( i = n - 1 \), this means that

\[
p_{n-2} > \frac{p_{n-1}}{1 + p_{n-1}},
\]

which implies that there is a strictly positive number of passengers coming to the station at \( t_{n-1} \).
Substituting (A.86) and (A.87) in the following expressions where the third line uses (A.84),

\[
E [c(t_{i-1})] = p_{i-1} c^{SP}(t_{i-1}) + (1 - p_{i-1}) (h \alpha + E [c(t_i)]) \tag{A.88}
\]

\[
E [c(t_{i-2})] = p_{i-2} c^{SP}(t_{i-2}) + (1 - p_{i-2}) (h \alpha + E [c(t_{i-1})]) \tag{A.89}
\]

\[
> p_{i-2} c^{SP}(t_{i-2}) + (1 - p_{i-2}) (h \alpha + E [c(t_i)]), \tag{A.90}
\]

implies that

\[
E [c(t_{i-2})] > E [c(t_{i-1})]. \tag{A.91}
\]

Applying this logic recursively eventually yields

\[
E [c(t_n)] > E [c(t_{n-1})].
\]

This, together with the result in (i), implies that

\[
E [c(t_1)] > E [c(t_{n-1})] \tag{A.92}
\]

which is a contradiction to the equilibrium. Therefore, for \(i \in \{n + 1, \ldots, l\}\),

we have that

\[
E [c(t_{i-1})] \leq E [c(t_i)]. \tag{A.93}
\]

Combining this with (A.83) yields the desired result. \(\blacksquare\)
A.3. Proof of Lemma 1

Lemma A.1 implies that for $i \in \{1, \cdots, R\}$, the expected trip cost is

$$E[c(t_i)] = \frac{1}{R-i+1} c^{SD}(t_i) + \frac{R-i}{R-i+1} (h\alpha + E[c(t_{i+1})]). \quad (A.94)$$

Solving this recursively and using that $E[c(t_R)] = c^{SD}(t_R)$ yields

$$E[c(t_i)] = c^{SD}(t_i) + (R-i) \frac{h\alpha + h\gamma}{2} \quad (A.95)$$

$$= c^{SD}(t_R) + (R-i) \frac{h\alpha - h\gamma}{2}; \quad (A.96)$$

therefore for the $l$th run it is

$$E[c(t_i)] = c^{SD}(t_R) + (R-l) \frac{h\alpha - h\gamma}{2}. \quad (A.97)$$

Lemma A.2 implies that none comes to the origin station after $t_n$, i.e., $D(t_n) = N$. Therefore, for $i \in \{n, \cdots, l-1\}$, the probability of boading the $i$th run is $1/(R-i+1)$ and the expected trip cost is

$$E[c(t_i)] = \frac{1}{R-i+1} c^{SD}(t_i) + \frac{R-i}{R-i+1} (h\alpha + E[c(t_{i+1})]). \quad (A.98)$$
Solving above recursively yields¹⁹

\[
E[c(t_i)] = \frac{l - i}{R - i + 1} \left[ c^{SD}(t_{l-1}) + \frac{l - i - 1}{2} h(\alpha + \beta) \right] \\
+ \frac{R - l + 1}{R - i + 1} [(l - i) h\alpha + E[c(t_i)]]
\]  
(A.99)

where \( E[c(t_i)] \) is given in (A.97). When \( i = n \) the above expression becomes

\[
E[c(t_n)] = \frac{l - n}{R - n + 1} \left[ c^{SD}(t_{l-1}) + \frac{l - n - 1}{2} h(\alpha + \beta) \right] \\
+ \frac{R - l + 1}{R - n + 1} [(l - n) h\alpha + E[c(t_i)]]
\]  
(A.100)

\[
= \frac{l - n}{R - n + 1} \left[ c^{SD}(t_{l-1}) + \frac{l - n - 1}{2} h(\alpha + \beta) \right] \\
+ \frac{R - l + 1}{R - n + 1} [(l - n) h\alpha + c^{SD}(t_R) + (R - l) \frac{h\alpha - h\gamma}{2}]
\]  
(A.101)

which is equal to \( \bar{c} \). Moving the entire scheduling \( t_1, \ldots, t_R \) a little bit earlier decreases the schedule delay cost for the last run and thus decreases the expected travel-time cost for the \( l \)th run given in (A.97) as well. Then, the expected trip

¹⁹Derivation is as follows: since for runs from the \( n \)th to the \( l - 1 \)th, the probability of boarding is given as \( 1/(R - i + 1) \). Thus we have

\[
E[c(t_{l-1})] = \frac{c^{SD}(t_{l-1}) + R - l + 1}{R - l + 2} (h\alpha + E[c(t_i)])
\]

\[
E[c(t_{l-2})] = \frac{c^{SD}(t_{l-2}) + R - l + 2}{R - l + 3} (h\alpha + E[c(t_{l-1})])
\]

\[
= \frac{c^{SD}(t_{l-1}) + c^{SD}(t_{l-2})}{R - l + 3} + \frac{2R - 2l + 3}{R - l + 3} h\alpha + \frac{R - l + 1}{R - l + 3} E[c(t_i)],
\]

and in general,

\[
E[c(t_i)] = \frac{c^{SD}(t_i) + \cdots + c^{SD}(t_{l-1})}{R - i + 1} + \frac{(l - i)(2R - l - i + 1)}{2(R - i + 1)} h\alpha + \frac{R - l + 1}{R - i + 1} E[c(t_i)].
\]

Since \( t_{l-1} < t^* \) we have the result.
cost for the $n$th run given in (A.101) will be lower, and so is $\tilde{c}$. Therefore, the equilibrium expected travel-time cost $\tilde{c}$ is minimized when the scheduling is at its earliest possible under the feasibility condition that $c^{SD}(t_1) \leq \tilde{c}$. Thus, in the second best, the equilibrium expected travel-time cost $\tilde{c}$ must be equal to the schedule delay cost of the first run.

\footnote{See Appendix B.6 for the proof.}