INTERNATIONAL UNIVERSITY OF JAPAN
Public Management and Policy Analysis Program
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Decision Analysis

Decision analysis is basically non-strategic, normative (as opposed to descriptive), static, discrete, and probabilistic. Decision strategies depend on the level of knowledge: certainty, risk, and uncertainty. The states of the Nature should be mutually exclusive and collectively exhaustive.[1]

Figure 1. Level of Knowledge: Certainty, Risk, and Uncertainty

		Probability		
		Determined (100%)	Known	Unknown
Possibility	Known	*Certainty*	*Risk*	*Uncertainty*
	Unknown		-	*Ignorance*

1. Decision-making under Certainty

Under certainty, decision-makers know all the possibilities (events or states of the Nature) and their outcomes are perfectly predicted. The comprehensive analytic decision-making assumes this circumstance. A decision-making will follow such steps as,

a. List all alternatives given a state of the Nature
b. List outcomes (costs and benefits) of all alternatives
c. Compare outcomes of alternatives
d. Choose the best alternative that has a maximum net value

Suppose you are the mayor of the City of Minami Uonuma and decide the best way to remove heavy snow. There are three alternatives of using sprinkler, snow blower, and Sodium Chloride; you need to choose one of them in order for simplicity although three options can be used together in reality.

Table 1. A Cost Benefit Analysis for Removing Snowfall

	Sprinkler	Snow Blower	Sodium Chloride
Benefits			
- Benefit item 1	…	…	…
- Benefit item 2	…	…	…
- …	…	…	…
Costs			
- Cost item 1	…	…	…
- Cost item 2	…	…	…
…	…	…	…
Net	♣	♥	♦

[1] If one of the states of the Nature occurs, it automatically implies that none of the other states occurs (mutually exclusive). In addition, one of the states must occur or the states defined include all possibilities (collectively exhaustive).

Since you know whether condition (when snow will fall and how much) and others perfectly (e.g., a snow blower can remove 10 tons of snow per hour), your job is to calculate the net of each alternative and choose the best one to remove snow on public areas.

2. Decision-making Under Risk

Under risk, decision-makers know all possibilities (outcomes) and their probabilities; they cannot predict possibilities (events) perfectly. Under this circumstance, the followings decision-making criteria are used to minimize risk.

2.1 The easiest way is to choose the **most probable state of the Nature (Maximum likelihood criterion)** and then compare alternatives under the most likely state only. Because of the higher likelihood of being dry (.75 > .25), Max will consider only payoffs for dry; he will sell the land to get 90 (90 > -100) assuming that there will be no oil (being dry) in that land (Table 2).

Table 2. Maximum Likelihood

	State of Nature	
	Oil (.25)	Dry (.75)
Drill	700	-100
Sell	90	90

Table 3. Expected Monetary Value

	State of Nature		EMV
	Oil (.25)	Dry (.75)	
Drill	700	-100	100
Sell	90	90	90

2.2 You may assign weight (likelihood or probability) to each state of the Nature and compute expected values (EV) like **Expected Monetary Value (EMV)**. This decision criterion is also called Bayes' decision rule. The alternative that has the highest expected profit (or the lowest expected cost) will be chosen. $EMV_i = \sum p_i r_i$, where r and p respectively represents the reward (payoff) of the ith alternative and its probability. If Max drills for oil, his EMV is 100 = .25 × 700 + .75 × (-100). When selling the land, he will receive 90 = .25 × 90 + .75 × 90. Hence, he will drill for oil (100>90). See Table 3.

2.3 The **Expected Opportunity Loss (EOL)** (or Expected Regret) is calculated using the amount of regret, which is determined by subtracting the alternative payoffs for each state of the Nature from the maximum payoff for that state. $EOL_i = \sum p_i l_i$, where l represents opportunity loss of the ith alternative. The alternative with the minimum expected regret will be chosen. Note that EMV and EOL always lead to the same choice; if not, you are off the track.

2.4 If there is oil in the land, the maximum payoff is 700 (700>90) and drilling for oil has zero regret (=700-700) because this option brings the maximum profit of 700. But selling the land comes with a regret of 610=700-90 (Max will receive 90 but lose the opportunity of earning the maximum 700 in case of oil in the land). If there is no oil, selling the land is the best option with zero regret (=90-90), while drilling leaves a regret of 190 = 90 – (-100) (Max will waste 100 for drilling cost and, in addition, lose the opportunity of earning the maximum 90 in case of no oil). See Table 5 below.

Table 4. Payoff Tables

Payoff	State of Nature	
	Oil (.25)	Dry (.75)
Drill	700	-100
Sell	90	90

Table 5. Expected Opportunity Loss

Regret	State of Nature	
	Oil (.25)	Dry (.75)
Drill	0=700-700	190=90-(-100)
Sell	610=700-90	0=90-90

2.5 If Max drills, his EOL is 142.5 = .25 × 0 + .75 × 190. If he sells the land, his EOL is 152.5 = .25 × 610 + .75 × 0. As a consequence, he will drills to get minimum EOL of 142.5 (142.5<152.5). Do not choose maximum EOL to "regret" much. See Table 6.

Table 6. Decision-making Using EOL

Payoff	State of Nature		Regret	State of Nature		EOL
	Oil (.25)	Dry (.75)		Oil (.25)	Dry (.75)	
Drill	700	-100	Drill	0	190	142.5
Sell	90	90	Sell	610	0	152.5

2.6 Table 7 below summarizes decision-making under risk. Keep in mind that EOL uses regret, NOT payoff; the smaller, the better.

Table 7. Decision-making Under Risk

Payoff	State of Nature		Maximum Likelihood	EMV	EOL
	Oil (.25)	Dry (.75)	(Payoff)	(Payoff)	(Regret)
Drill	700	-100	-100	100	142.5
Sell	90	90	90	90	152.5

3. Decision-making Under Uncertainty

Under uncertainty, decision-makers know all possibilities but do not know their probabilities of possible events. Then, how do they make a decision? The following criteria are used for decision-making under uncertainty.

3.1 The **pessimistic criterion (MAXIMIN)** or Wald criterion, suggested by Abraham Wald, maximizes the minimum payoff. This criterion assumes that the worst will always happen no matter what alternative you select. From the conservative view, you estimate the minimum returns for alternatives and pick up the maximum of them or the "**best of the worst**." Thus, it provides a way of avoiding the worst outcome. If Max drills, the worst state of nature is that the land does not have oil; he will just lose 100. If he sells the land, he will get 90 regardless of the state of nature. Therefore, Max will sell the land (90 > -100).

3.2 The **optimistic criterion (MAXIMAX or MINIMIN)** chooses the alternative with maximum of maximum payoff. This criterion assumes that the best possible outcome will always occur for any choice you make. This criterion chooses the "**best of the best**." But it does not provide protection against the potentially worst outcomes. If Max drills, the best scenario will occur and give him 700. If he sells the land, he will get 90 regardless of the state of nature. As a consequence, he will drill for oil to get 700 (700 > 90).

3.3 **Laplace criterion (Equal likelihood criterion)** computes the expected values using equal probability for each state of nature. It assumes that all states of nature are equally likely to occur. Then choose the alternative with maximum expected profit (or minimum expected cost). Laplace criterion is similar to EMV (expected monetary value) except for equal probabilities assigned. When Max drills for oil, he is expected to get 300 = 700 ×.5 + (-100) ×.5. Note that the equal probability .5 is assigned to each state (oil or no oil). If he sells the land, the expected profit is 90 = 90 ×.5 + (90) ×.5. Therefore, he will drill to get the expected monetary value of 300 (300>90).

3.4 **Hurwicz criterion (realism)**, suggested by Leonid Hurwicz, computes the weighted sum of the optimistic (Maximum) and pessimistic (Minimum) evaluations using the

coefficient of optimism *α*: $Hurwicz_i = \alpha \times Max_i + (1-\alpha) \times Min_i$. The coefficient of optimism lies between zero (completely pessimistic) and one (completely optimistic). Then, choose an alternative with the maximum of the weighted sum. It combines MAXIMIN and MAXIMAX criteria. But it is difficult to determine the appropriate *α*, since it varies from person to person. While EMV and Laplace take all states of the Nature into account, Hurwicz criterion considers only maximum and minimum payoffs of the alternatives regardless of the number of states. If *α* =.2, drilling will give him 60 = .2 × 700 + (1-.2) × (-100), whereas selling the land will bring 90 = .2 × 90 + (1-.2) × (90). Therefore, Max will sell the land (90 > 60).

3.5 **Savage criterion of regret (MINIMAX Regret)**, suggested by L. J. Savage, is based on the opportunity loss under risk. It minimizes the largest anticipated regret. The concept regret explained in EOL remains unchanged, but Savage does not use probabilities of states; you don't know probabilities under uncertainty. Once the regret table is made, choose the minimum of the maximum regret. From the above regret table, the maximum regret of drilling is 190 (190>0) and that of selling is 610 (610>0). Therefore, Max will drill the land to minimize regret (190 < 610). Table 8 summarizes decision-making under uncertainty.

Table 8. Decision-making Under Uncertainty

Payoff	State of Nature		MaxiMin	MaxiMax	Laplace	Hurwitcz	MiniMax
	Oil	Dry				(*α* = .2)	(Regret)
Drill	700	-100	-100	700	300	60	190
Sell	90	90	90	90	90	90	610

3.6 Since people tend to be somewhat pessimistic, pessimistic (MAXIMIN) and Savage (MINIMAX) criteria are most popular in practice. The former makes sure that the worst things that can happen are not too bad, whereas the latter avoids the potential worst using post hoc evaluation of outcomes. By the way, difficulty determining the coefficient of pessimism makes Hurwicz criterion impractical.

* What is the value of knowing decision criteria? You may be able to understand the underlying decision criteria that your counterparts in negotiation have in their mind; you will be better off by taking advantage of such understandings.

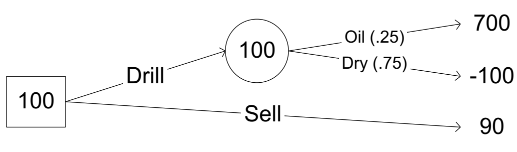
4. Decision Tables versus Decision Trees

A decision table (payoff table) can be converted to its corresponding decision tree and vice versa. A decision tree is a graphic tool for showing the sequence of decision alternatives and events in a decision situation. In addition to H&H, you may also read Albright & Winston (2006) chapter 8 and Hillier & Lieberman (2010) chapter 15.

Table 9. A Decision Table

Payoff	State of Nature	
	Oil	Dry
Drill	700	-100
Sell	90	90

Figure 2. A Decision Tree



A decision table can summarize many pieces of information in a succinct table form, but it is a bit difficult for decision makers to grasp the key idea of the decision situation they are encountering. By contrast, a decision tree makes it easy and intuitive to understand the decision situation. However, a decision tree will be problematic when there are many alternatives and states of the Nature.

For instance, a 20 by 10 decision table can summarize a decision situation where there are 20 alternatives and 10 different states of nature. However, this decision tree needs 200 (=20 × 10) final branches! Therefore, decision trees are appropriate in such situations as where decision-making is sequential and where there are a small number of alternatives and states of the Nature.

5. Sequential Decision-making (Decision Tree)

A decision tree is a graphic tool to show the sequences of alternatives and events involved.

Decision trees assume that

a. All the necessary alternatives are known
b. There should be only limited options and limited states of the Nature
c. Concrete quantitative or monetary information should be available to make a decision
d. The probability and payoff for each choice are also known.

A decision tree consists of

a. *Decision nodes* (decision points or decision forks) represented by a square or a box □,
b. *Chance nodes* (chance points or event forks) represented by circles ○,
c. *Alternatives* represented by branches (lines linking decision nodes and chance nodes),
d. *States of the Nature* (mutually exclusive and collectively exhaustive), and
e. *Payoffs* of individual alternatives.

Figure 3. A Good Decision Tree

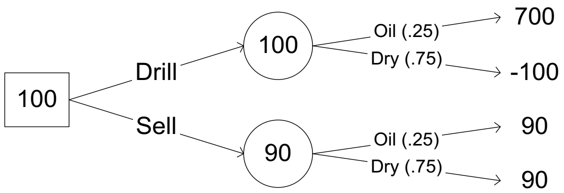
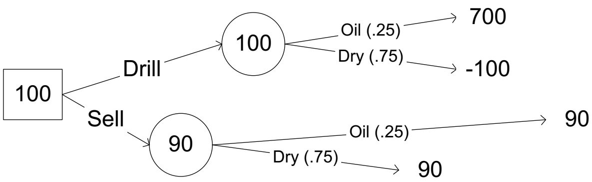


Figure 4. A Bad Decision Tree



Oftentimes decision trees implicitly contain a time frame beginning from the left to the right. Therefore, decision nodes and chance nodes at the same time point should be vertically aligned properly (Figure 3). Figure 4 implies that the Nature decides twice consecutively: one after Max decides to sell the land and then another longer time after drilling for oil. Yes, this interpretation does not make any sense. Similarly, payoffs should be located at the very end (to the right) of a decision tree and vertically aligned. Do not draw other symbols and leave any scribble or computation on a decision tree. Compare Figure 3 and 4 very carefully.

Figure 5. You Decide First

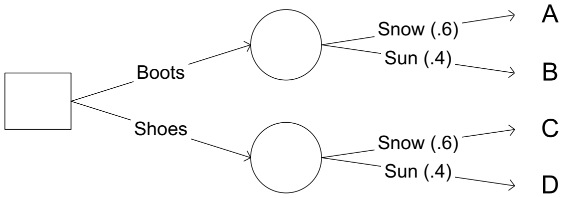
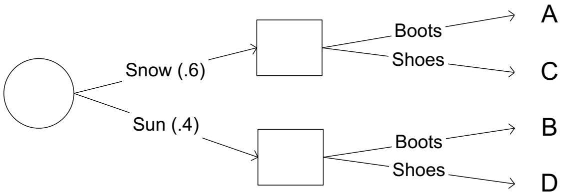


Figure 6. Nature Decides First



You MUST distinguish a square (decision nodes) from a circle (chance nodes); your decision is absolutely different from Nature's decision. Otherwise, you are saying that the Nature decides whether or not to wear boots and you will select either sunny or snowy day!!!

In general, there are three possibilities:

a. A decision tree begins with decision nodes (you decide first) and then the Nature decides. See Figure 5;
b. The Nature decides first and then you decide. See Figure 6; and
c. You (player 1) decide first and then other person (player 2) decides. There is no Nature and box in a decision tree.

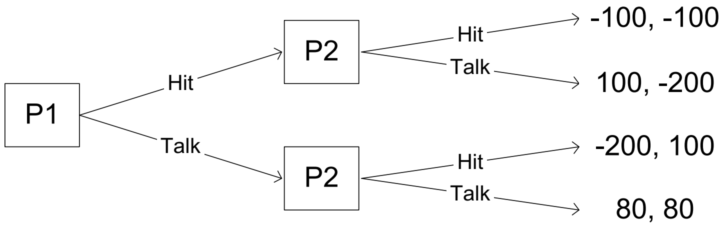
It is absurd, however, to imagine a situation where the Nature 1 decides first and then the Nature 2 decides since the Nature, heaven, or god is supposed to know everything and accordingly one decision-making will do.

The last case is a (strategic) game. Imagine you are playing chess with one of your friends. The following is an extensive form of a game. Note that there is no circle for chance node and payoffs are paired (first payoff for player 1 and the second for player 2). If player 1 hits player 2 who talks to (rather than hits players 1 back) player 1 as response, for instance, player 1 will get 100 and player 2 will get -200. Given this circumstance, what is the optimal strategy for each of the players? How do we know that? *Backward induction* is the answer.

Table 10. A Payoff Table

	Player 2	
Player1	Hit	Talk
Hit	(-100,-100)	(100,-200)
Talk	(-200,100)	(80,80)

Figure 7. Extensive Form (Decision Tree) of a Game

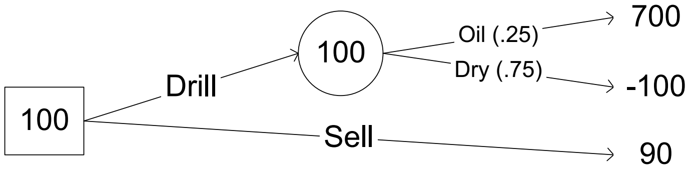


Alternatives need to be labeled appropriately; otherwise, you do not know which alternative is which. At the very ends of alternatives, you should provide payoffs of the alternatives. Do not arbitrarily change or add alternatives. In the Warren Buffy case, for instance, some of you might consider "no investment" and get stuck. Huh… "No investment" is not defined and thus it is not his alternative at all! You are just off the track! Please calm down and do not worry about problems that might happen in Andromeda Galaxy when analyzing a decision; Darth Vader and his son Luke would take care of Andromeda problems for you!

Alternatives immediately under a chance point should be mutually exclusive and collectively exhaustive. *Mutually exclusive* means that one state should be clearly distinguished from other states. If you define three states of weather as "sunny," "rainy," and "snowy," "sunny" does not, for example, include entire or part of "rainy" or "snowy." *Collectively exhaustive* means that states you listed include all possibilities of the Nature you defined; that is, the sum of all probabilities must be one (otherwise, that is not, by definition, a probability distribution). It will be problematic if you define three types of weather and then assign probabilities, say, .5, .2, and .2, absurdly, reserving the remaining .1 to "cloudy." Please avoid this inconsistent and absent-minded behavior.

In order to determine the optimal decision in a decision tree, *work backward from the right to the left.* Evaluate chance points using expected monetary values (EMV). You may include probabilities of alternatives on the branch labels and EMV in a box or circle. They are informative (and recommended) but not strongly required.

Figure 8. Max's Decision Tree



EMV of drilling: 100 = .25 × 700 + .75 × (-100).
EMV of selling: 90 = .25 × 90 + .75 × 90. = 1.0 × 90

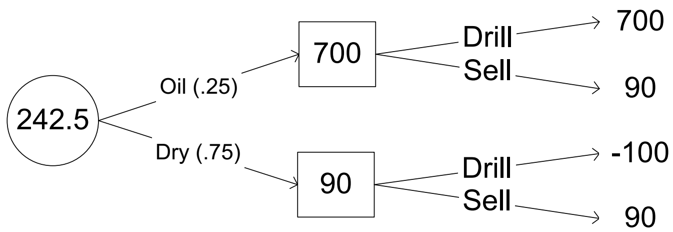
In the decision tree above, EMV of drilling 100 is assigned to the circle (chance node) and EMV 100 again is assigned to the square (decision node) since EMV of drilling is greater than EMV of selling. Therefore, the decision is drilling (100>90).

6. Expected Value of Perfect Information (EVPI)

What is the value of information in decision-making? If the benefits of additional information offset the acquiring costs, you would like to get the information. There are two kinds of information: *perfect information* (the future state of the Nature can be predicted exactly) and *imperfect information*.

6.1 If perfect information is available, you will know which states of nature will occur before making your decision. Consequently, *perfect information eliminates all regrets.*

Figure 9. A Decision Tree Under Perfect Information



6.2 <u>Perfect information about the future alters the orders of events</u>. Perfect information reorders the decision tree so that a decision tree begins with event forks (chance nodes). Please pay special attention to the order of circle and box (square) in Figure 8 and 9. In this circumstance, you know if the land has oil or not before deciding to drill or sell. Don't be misled by the probabilities presented in event forks in Figure 9; the Nature remains unchanged, whereas your perfect information enables you to know Nature's behavior (oil or dry) perfectly.

6.3 If you know the presence of oil in advance, obviously you will drill to get the profit of 700; only idiots will sell the land in case of oil in the land. If you know no oil exists in the land, you will sell the land and avoid waste of -100. As a result, you can see 700 and 90 in the boxes (squares) in Figure 9. Expected monetary value with perfect information is 242.5 = .25 × 700 + .75 × 90.

6.4 The *expected (monetary) value of perfect information* (EVPI) is the difference between the expected value (i.e., profit or cost) with and without perfect information. EVPI = [Expected value with perfect information] – EMV. In the example, EVPI = 142.5 = 242.5 - 100.

6.5 Using perfect information, you can always select zero-regret alternatives. Therefore, EVPI=EOL. Compare the EVPI with EOL in 2.5 to make sure this conclusion.

6.6 EVPI sets the maximum value that information can have; that is, EVPI provides the upper bound on the value of information; any information can NEVER be worth more than EVPI in this particular setting.

6.7 In reality, it is not realistic for you to be able to obtain perfect information (less practical). Consequently, a very common use about EVPI is as a preliminary screening tool for obtaining more information.

6.8 For example, if someone suggests purchasing a piece of perfect information for 150 (>142.5), he/she is trying to rip you off! Don't be fooled by such a monkey business! Oftentimes many stupid public managers are ripped off by such a wicked broker and waste a huge amount of budget. Remember that no information is more informative than perfect information.

7. Expected Value of Imperfect Information (EVII)

What if you have additional information that is not perfect (some parts of the information are correct, while others are not)? What is the value of such imperfect information (or sample information)? How to calculate it? Is it worthwhile to obtain such imperfect information?

7.1 Imperfect information allows you to have a better "guess" about what will happen. You will be able to refine the probabilities about the future states of the Nature somewhat but will not be able to predict future states perfectly. Imperfect information by definition involves some random components (incorrect information).

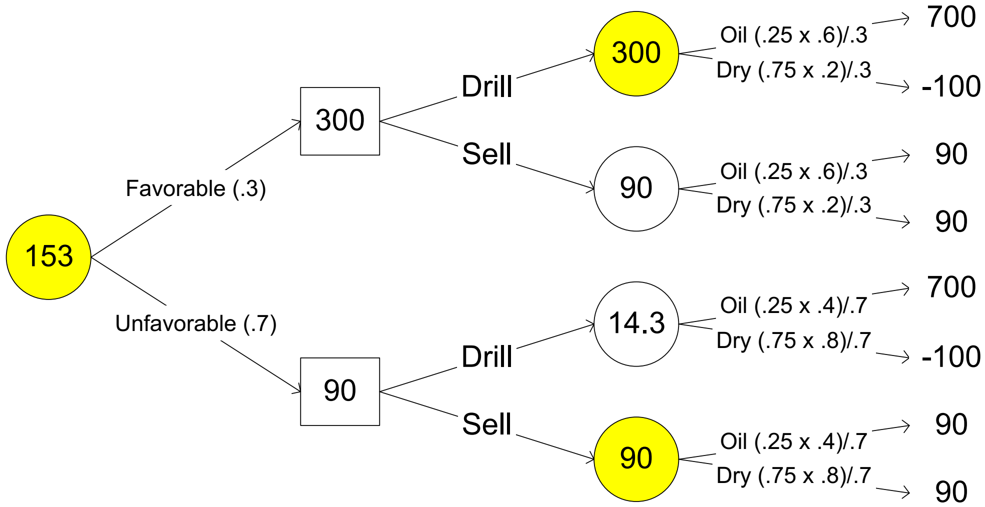
7.2 Where and how can you get *imperfect information*? An effort to get information (e.g., experimentation or survey) or empirical evidence will provide some pieces of information with some errors that are rarely perfect information. The seismic survey in the example provides imperfect information in the conditional probability form. Note that perfect information Never, Ever makes any mistake.

7.3 The following seismic survey information is not perfect; P(Favorable|Oil) and P(Unfavorable|Dry) are correct but P(Unfavorable|Oil) and P(Favorable|Dry) are incorrect.

- P(Favorable|Oil)=.6 : probability of favorable seismic soundings given oil land
- P(Unfavorable|Oil)=.4 : probability of unfavorable soundings given oil land
- P(Favorable|Dry)=.2 : probability of favorable seismic soundings given dry land
- P(Unfavorable|Dry)=.8 : probability of unfavorable soundings given dry land
- Notice that, by definition, P(Favorable|Oil)+P(Unfavorable|Oil)=1 and P(Favrable|Dry)+P(Unfavorable|Dry)=1.

7.4 A decision trees under imperfect information tends to be messy. Let us now focus the structure of the decision tree. The story goes like this. First, you get the result of the seismic survey (either favorable or unfavorable). This is a chance node that you cannot control. Based on this result, you decide to drill or sell the land. Then, another chance node (God or heaven) selects oil or dry.

Figure 10. A Decision Tree Under Imperfect Information



7.5 The key question is how informative or valuable this imperfect information is in your decision-making. This concept is called the *expected (monetary) value of imperfect information* (EVII) or EV of experimentation or EV of sample information. EVII is calculated as [Expected value with imperfect information] – EMV.

7.6 EVII uses a concept of conditional probability and specifically Bayes' theorem (rule) that involves joint probabilities, marginal probabilities, and eventually posterior probabilities. In the decision tree above, P(Favorable)=.3 and P(Unfavorable)=.7 are *marginal probabilities* and P(Oil|Favorable) = (.25 × .6)/.3 = .5, P(Oil|Unfavorable) = (.25 × .4)/.7 = 1/7, and others are *posterior probabilities*. The critical issue is how to calculate posterior probabilities from prior and conditional probabilities (given in 7.3).

7.7 These posterior probabilities or updated conditional probabilities are applied to calculation of expected value with imperfect information.

- Conditional probability: $P(B\mid A)=\dfrac{P(A\cap B)}{P(A)}$
- Bayes' Theorem (rule):

$$P(B_*\mid A_*)=\frac{P(B_*\ \&\ A_*)}{P(A_*)}=\frac{P(A_*\mid B_*)P(B_*)}{\sum P(A_*\mid B_i)P(B_i)}=\frac{P(A_*\mid B_*)P(B_*)}{P(A_*\mid B_1)P(B_1)+P(A_*\mid B_2)P(B_2)+\ldots}$$

7.8 Let us check the information available now. First, we have a common knowledge called *prior probabilities*: P(oil)=.25 and P(dry)=.75. These prior probabilities come from experiences and past records without imperfect information. Second, we also have additional, imperfect, information or empirical evidence (seismic survey in this example) in a form of *conditional probabilities* (because we don't know perfectly): P(Favorable|Oil)=.6, P(Unfavorable|Oil)=.4, P(Favorable|Dry)=.2, and P(Unfavorable|Dry)=.8.

7.9 Our goal is to get such posterior probabilities as P(Oil|Favorable), P(Oil|Unfavorable), P(Dry|Favorable), and P(Dry|Unfavorable) from two sets of information (prior and additional information). In order to get P(Oil|Favorable), for instance, we need to know a joint probability P(Oil & Favorable) and a marginal probability P(Favorable) because $P(Oil\mid Favorable)=\dfrac{P(Oil\ \&\ Favorable)}{P(Favorable)}$. P(Oil & Favorable) is obtained from an additional conditional probability P(Favorable|Oil) and a prior probability

P(Oil) that are given in question since $P(Favorable \mid Oil) = \frac{P(Oil \,\&\, Favorable)}{P(Oil)}$. A marginal probability P(Favorable) is the sum of P(Oil & Favorable) and P(Dry & Favorable).

7.10 Now let us start calculation. We need to get the numerator of the Bayes' formula. This is *joint probability*.
- P(Oil & Favorable)= P(Favorable|Oil) × P(Oil) = .6 × .25 = .15
- P(Dry & Favorable) = P(Favorable|Dry) × P(Dry) = .2 × .75 = .15
- P(Oil & Unfavorable) = P(Unfovaorable|Oil) × P(Oil) = .4 × .25 = .1
- P(Dry & Unfavorable) = P(Unfavorable|Dry) × P(Dry) = .8 × .75 = .6

7.11 Next step is to get the denominator of the Bayes' formula. This is (*unconditional) marginal probability*.
- P(Favorable seismic soundings) = P(Oil & Favorable) + P(Dry & Favorable) = .15 + .15 = .3.
- P(Unfavorable seismic soundings) = P(Oil & Unfavorable) + P(Dry & Unfavorable) = .1 + .6 = .7.
- By definition, P(Favorable) + P(Unfavorable) = 1.

7.12 Then, obtain *posterior probabilities* (updated conditional probabilities by calculating the numerator (joint probability) divided by the denominator (marginal probability)
- **Posterior probability = (joint probability) / (marginal probability)**
- P(Oil|Favorable) = P(Oil & Favorable)/P(Favorable) = .15/.3= .5.
- P(Dry|Favorable)=P(Dry & Favorable)/P(Favorable) = .15/.3 = .5.
- P(Oil|Unfavorable) = P(Oil & Unfavorable)/P(Unfavorable) = .1/.7= 1/7.
- P(Dry|Unfavorable)= P(Dry & Unfavorable)/P(Unfavorable) = .6/.7 = 6/7.
- By definition, P(Oil|Favorable)+P(Dry|Favorable)=1 and P(Oil|Unfavorable) + P(Dry|Unfavorable)=1

7.13 Then, calculate the expected monetary value of each alternative given favorable and unfavorable seismic soundings (imperfect information). You must use the posterior probabilities (updated conditional probabilities) instead of prior probabilities.
- EV of drilling given the favorable is 300 = (.5 × 700) + .5 × (-100)
- EV of selling given the favorable is 90 = (.5 × 90) + (.5 × 90)
- EV of drilling given the unfavorable is 14.2857 = (1/7) × 700 + (6/7) × (-100)
- EV of selling given the unfavorable is 90 = (1/7) × 90 + (6/7) × 90

7.14 If you get favorable seismic soundings (30 percent chance=.3), you will be better off by drilling the land since EV of drilling given favorable soundings (300) is larger than EV of selling (90): 300>90. In case of unfavorable seismic soundings (70 percent chance=.7), selling the land will give you better expected value of 90, which is higher than 14.2857 that you could get by drilling for oil: 90>14.2857

7.15 Calculate expected values with imperfect information: 153 = .3 × 300 + .7 × 90. Drilling given favorable soundings and selling given unfavorable soundings are considered in calculation. Notice that marginal probabilities (not prior probabilities) are used in this calculation.

7.16 Finally, calculate the expected value of imperfect information (EVII): 53 = 153 - 100. Make sure that EVII is always smaller than or equal to EVPI (53<142.5).

7.17 As far as EVII is larger than the cost of the imperfect information, the information is worthwhile; otherwise, you will be worse off by wasting money for such crappy information. In the example, EVII of 53K is larger than the cost of 30K. As a consequence, the seismic survey is worth 23K (=53-30) which is net benefit of utilizing the seismic survey.

Appendix 1: Bayes' Theorem

If observable events (from empirical evidence or additional information) $A_1, A_2, ..., A_e$ are mutually exclusive, and states of nature $B_1, B_2, ..., B_n$ are also mutually exclusive, the *posterior probabilities* (*prior probabilities* updated after observing the event B_*) can be calculated using observable events as follows.

$$P(B_* \mid A_*) = \frac{P(B_* \,\&\, A_*)}{P(A_*)} = \frac{P(A_* \mid B_*)P(B_*)}{\sum P(A_* \mid B_i)P(B_i)} = \frac{P(A_* \mid B_*)P(B_*)}{P(A_* \mid B_1)P(B_1) + P(A_* \mid B_2)P(B_2) + ... + +P(A_* \mid B_n)P(B_n)}$$

where A_* and B_* respectively denote a particular observable event and a particular state of nature that you are interested in.

An example here comes from R. Lyman Ott's *An Introduction to Statistical Methods and Data Analysis* (Belmont, CA: Duxbury Press, 1993: 138). The probability that a citizen has tuberculosis (TB) is known to be .001, P(TB)=.001 and P(Not TB)=1-.001=.999. A reliable screening test can detect 95 percent of TB correctly; 95 percent of citizens with TB will show a positive result in the test. Only two percent of those who don't have TB will show a positive result. Of course, this test is not perfect. Empirical evidence or additional information is summarized as following conditional probabilities.

P(Positive|TB) = .95
P(Negative|TB) = 1 - .95 = .05
P(Positive|No TB) = .02
P(Negative|No TB) = 1 - .02 = .98

Prior probabilities are P(TB)=.001 and P(No TB)=.999.

Possible questions are, "What is the probability that a citizen has TB if his test result is positive?" P(TB|Positive), "What is the probability that a citizen has TB, given a negative test result?" P(TB|Negative), and so on. These questions ask, "To what extent can we improve our prediction if we use additional *imperfect* information?" We can apply Bayes' theorem to answer such questions.

$$P(TB \mid Positve) = \frac{P(TB \,\&\, Positive)}{P(Positive)} = \frac{P(Positive \mid TB)P(TB)}{P(Positive \mid TB)P(TB) + P(Positive \mid NoTB)P(NoTB)} = \frac{.95 \times .001}{.95 \times .001 + .02 \times .999} = .045$$

$$P(TB \mid Negative) = \frac{.05 \times .001}{.05 \times .001 + .98 \times .999} = .0005$$

$$P(NoTB \mid Positive) = \frac{.02 \times .999}{.95 \times .001 + .02 \times .999} = .955 = 1 - .045$$

$$P(NoTB \mid Negative) = \frac{.98 \times .999}{.05 \times .001 + .98 \times .999} = .9995 = 1 - .0005$$

Since a positive (or negative) result means that a citizen can be either TB or No TB, the sum of P(TB|Positive) and P(No TB|Positive) must be 1: P(TB|Positive) + P(No TB|Positive) = 1. Similarly, P(TB|Negative) + P(No TB|Negative)=1.

This calculation seems a bit messy to beginners. Calculation of such posterior probabilities can be broken into several steps.

First calculate joint probabilities. P(TB) and P(No TB) are given as prior probabilities, while P(Positive|TB), P(Negative|TB), P(Positive|No TB), and P(Negative|No TB) are provided as conditional probabilities (empirical evidence) from the screening test. Notice that these joint probabilities are *numerators* in the Bayes' formula.

P(TB & Positive) = P(Positive|TB)P(TB) = .95×.001 = .00095
P(No TB & Positive) = P(Positive|No TB)P(No TB) = .02×.999 = .01998
P(TB & Negative) = P(Negative|TB)P(TB) =.05×.001 = (1-.95) ×.001 = .00005
P(No TB & Negative) = P(Negative|No TB)P(No TB) = .98×.999 = (1-.02) ×.999 = .97902

Notice that, by definition, P(TB & Positve) + P(No TB & Positive) = 1 and P(TB & Negative) + P(No TB & Negative) = 1.

Next, calculate marginal probabilities of getting positive and negative results, respectively. Notice that these marginal probabilities are *denominators* in the Bayes' formula.

P(Positive) = P(Positive & TB) + P(Positive & No TB) = .00095 + .01998 = .02093
P(Negative) = P(Negative & TB) + P(Negative & No TB) = .00005 + .97902 = .97907
Notice that P(Positive) + P (Negative) = .02093 + .97907 = 1.

	Seismic Survey for Oil	Screening Test for Tuberculosis (TB)
Prior probability (Common Knowledge) ↓	P(Oil) = .25 P(Dry) = .75	P(TB) = .001 P(No TB) = .999
Conditional probability (Empirical Evidence or Additional Information) ↓	P(Favorable | Oil) = .6 P(Unfavorable | Oil) = .4 P(Favorable | Dry) = .2 P(Unfavorable | Dry) = .8	P(Positive | TB) = .95 P(Negative | TB) = .05 P(Positive | No TB) = .02 P(Negative | No TB) = .98
Joint probability (Numerator) ↓	P(Oil & Favorable) = .15 P(Dry & Favorable) = .15 P(Oil & Unfavorable) = .1 P(Dry & Unfavorable) = .6	P(TB & Positive) = .00095 P(No TB & Positive) = .01998 P(TB & Negative) = .00005 P(No TB & Negative) = .97902
Marginal probability (Denominator) ↓	P(Favorable) = .3 P(Unfavorable) = .7	P(Positive) = .02093 P(Negative) = .97907
Posterior probability (Updated conditional probability)	P(Oil | Favorable) = .5 P(Dry | Favorable) = .5 P(Oil | Unfavorable) = 1/7 P(Dry | Unfavorable) = 6/7	P(TB | Positive) = .045 P(No TB | Positive) = .95 P(TB | Negative) = .00005 P(No TB | Negative) = .99995

Finally calculate posterior probabilities using joint probabilities and marginal probabilities that you calculated above. That is, a posterior probability is corresponding joint probability divided by marginal probability.

P(TB|Positive) = P(TB & Positive)/P(Positive) = .00095/.02093 = .045
P(No TB|Positvie) = P(No TB & Positive)/P(Positive) = .01998/.02093 = .95 = 1-.045
P(TB|Negative) = P(TB & Negative)/P(Negative) = .00005/.97907 = .00005
P(No TB|Negative) = P(No TB & Negative)/P(Negative) = .97902/.97907 = .99995 = 1-.00005

Notice that, by definition, P(TB|Positve) + P(No TB|Positive) = 1 and P(TB|Negative) + P(No TB|Negative) = 1.

If you are quite familiar with Bayes' theorem (rule), apply it directly to get posterior probabilities. Otherwise, follow the three consecutive steps to calculate joint probabilities, marginal probabilities, and then posterior probabilities. The calculation process is summarized in the above table.

The key feature of Bayes' theorem is that you can update prior probabilities using empirical evidence or additional information to get posterior probabilities. You can get more accurate probabilities as far as the information is correct and thus reduce the risk of being wrong. The question is, "What is the value of such additional information to reduce the risk?"

Appendix 2: SUMPRODUCT Function in Excel
Excel's built-in function SUMPRODUCT (also in Quattro Pro) conducts "element-by-element" multiplication and then sum them up. Without this function, the multiplication (e.g., multiplication of coefficients and decision variables) will be painful especially when you have a large number of decision variables. The usage is SUMPRODUCT($A_{n\times m}$, $B_{n\times m}$, $C_{n\times m}$, …). This condition is the same as conformability of matrix addition and subtraction.

Look at the following Excel Worksheet for Max's decision-making. In order to calculate EMV for drilling, locate your cell pointer to E12 (as shown in the Figure) and provide the following formula "(B7*B9)+(C7*C9)." Similarly, provide "(B8*B9)+(C8*C9) in cell E13 for EMV for selling the land. Alternatively, you may provide SUMPRODUCT(B7:C7, B9:C9) in E12 and SUMPRODUCT(B8:C8, B9:C9) in E13 to get the identical result.

	A	B	C	D	E
6		Oil	No Oil		
7	Drill for oil	700	-100		
8	Sell the land	90	90		
9	Prior probabilities	0.25	0.75		
10					
11	Under Risk				
12	EMV for drilling = (700*.25) + (-100)*.75				100
13	EMV for selling = (90*.25) + (90*.75)				90
14	Choice under risk based on EMV				100

End of this document.