

INTERNATIONAL UNIVERSITY OF JAPAN
Public Management and Policy Analysis Program
Graduate School of International Relations

DCC5350 (2 Credits)
Public Policy Modeling
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This note is based on David Good's lecture, Hillier and Hillier (2014), Winston (2004), Albright and Winston (2005), and Hillier and Lieberman (2010). This note SHOULD NOT be used as a substitute of the textbook.

Linear Programming Interpretation

1. Interpretation of the Answer Report

The optimal solution is a set of final values of decision variables in the final tableau of the simplex method and can maximize or minimize the objective function.

1.1 *Optimal solution.* Read the optimal solution (a set of values of decision variables) under the label "Final Value" of the *Variable Cells* section in Table 1 below: $x_1=2$ and $x_2=6$. Cells \$C\$12 and \$D\$12 contain the values of x_1 and x_2 . You may ignore the values under "Original Value" since they are initial values used in the optimization (simplex method).

Table 1. Excel Solver's Answer Report

Objective Cell (Max)

Cell	Name	Original Value	Final Value
\$F\$12	Objective function	\$3,150	\$3,600

Variable Cells

Cell	Name	Original Value	Final Value	Integer
\$C\$12	x1	3	2	Contin
\$D\$12	x2	4.5	6	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
\$C\$14	Constraint 1 x1	2	\$C\$14<=\$F\$14	Not Binding	2
\$C\$15	Constraint 2 x1	12	\$C\$15<=\$F\$15	Binding	0
\$C\$16	Constraint 3 x1	18	\$C\$16<=\$F\$16	Binding	0

Maximize $y = 300x_1 + 500x_2$

subject to

$$x_1 \leq 4$$

$$2x_2 \leq 12$$

$$3x_1 + 2x_2 \leq 18$$

$$x_1 \geq 0, x_2 \geq 0$$

1.2 *Optimal value of the objective function.* Read the optimal value (maximum profit or minimum cost) of the objective function under "Final Value" of the *Objective Cell (Max)* section: P=3,600. Remember that cell \$F\$12 in the Excel worksheet contains the

mathematical expression of the objective function. Again you may ignore the values under “Original Value.”

1.3 Table 1 suggests that Wyndor Glass Co. produces two doors and six windows in order to maximize its profit up to $\$3,600 = \$300 \times x_1 + \$500 \times x_2 = \$300 \times 2 + \$500 \times 6$.

1.4 *Resources consumed to get the optimal value.* When producing two doors and six windows, Wyndor Glass Co. consumes two hours in plant 1: $2 = 1 \times 2$; 12 hours in plant 2 ($12 = 2 \times 6$); and 18 hours in plant 3 ($18 = 3 \times 2 + 2 \times 6$). These LHS values at the optimal solution are found under “Cell Value” of *Constraints*. Its calculation is summarized as:

- Plant 1: $2 = 1 \times x_1 + 0 \times x_2 = 1 \times 2 + 0 \times 6$
- Plant 2: $12 = 0 \times x_1 + 2 \times x_2 = 0 \times 2 + 2 \times 6$
- Plant 3: $18 = 3 \times x_1 + 2 \times x_2 = 3 \times 2 + 2 \times 6$

2. Characteristics of Constraints

A constraint is either binding or non-binding. Some constraints are superfluous or redundant.

2.1 *Slack* and *surplus* are the difference between LHS and RHS. Therefore, slack or surplus is defined as $|\text{RHS}-\text{LHS}|$, which is always nonnegative. Slack is calculated for a “ \leq type” constraint of resource limitation, while surplus is for a “ \geq type” constraint.¹ Slack describes the amount of unutilized resources (RHS-LHS), whereas surplus describes the amount by which a minimum requirement has been exceeded (LHS-RHS). Wyndor Glass Co. has a product mix problem with “ \leq type” constraints and thus has slack.

2.2 Let us calculate the slack of each constraint. The values of LHS are calculated in 1.4 above and the values of RHS are set by the policy problem. For instance, the slack of constraint 1 (plant 1) is calculated as $2 = 4$ (RHS: maximum number of hours available in plant 1) $- 2$ (LHS: labor hours utilized in plant 1 at the optimal solution). A slack of 2 means that the resource in plant 1 is not fully utilized; 2 hours remain unused in plant 1. By contrast, plant 2 and 3 utilize all available resources at the optimal solution and accordingly have zero slack. Check slacks under “Slack” of *Constraints* in Table 1.

- Slack in Plant 1: $2 = |\text{RHS}-\text{LHS}| = 4 - (1 \times 2) = 4 - 2$
- Slack in Plant 2: $0 = |\text{RHS}-\text{LHS}| = 12 - (2 \times 6) = 12 - 12$
- Slack in Plant 3: $0 = |\text{RHS}-\text{LHS}| = 18 - (3 \times 2 + 2 \times 6) = 18 - 18$

2.3 If the slack (or surplus) of a constraint is zero, the constraint is called a binding constraint (plant 2 and 3). Otherwise, the constraint is called a non-binding constraint (plant 1). Excel Solver displays binding or non-binding status under “Status” of *Constraints*; Constraints 2 and 3 in Table 1 are marked binding and their slacks are all zero.

2.4 *Meanings of binding constraints* Binding constraints form the boundary of the feasible set (feasible region) and thus limit the optimal solution. The optimal solution lies right on these binding constraint lines. The resources of binding constraints are fully utilized to have zero slack or surplus. Therefore, binding constraints are bottle necks or primary

¹ Albright & Winston (2005) do not distinguish surplus from slack. Excel Solver indicates slack or surplus under the Slack column. If a constraint has \leq in the Formula column and a positive value in the Slack column, it is slack; if a constraint has \geq and a positive value, it is surplus.

limiting factors in the optimal solution. Changing these constraints (i.e., RHS values) might produce a different solution to the problem. For instance, if the maximum number of hours available in plant 2 increases from 12 to 15, Wyndor Glass Co. will produce more door or window and thus get more profit. Therefore, binding constraints are the most important to public managers (decision makers).

- 2.5 *Meanings of non-binding constraints* In a non-binding constraint, NOT all resources are used up and some resources are left over at the optimal solution. The slack or surplus of a non-binding constraint is greater than zero (or positive). The constraint of plant 1 is non-binding because of its non-zero slack. Even if the maximum number of hours in plant 1 increases from current 4 to 10 hours, the optimal solution remains unchanged since the optimal solution needs only 2 hours in plant 1. Investment to increase resources (RHS) of non-binding constraints is useless and waste of money.
- 2.6 *Superfluous constraints* never touch the feasible set, whereas *redundant constraints* contain exactly the same limitation information as another constraint. Superfluous and redundant constraints are not important, if not useless, because ignoring them would lead to exactly the same choice. Wyndor Glass Co. case does not have such constraints.

3. Problematic Solutions

If a LP problem does not have a *unique solution* (single optimal solution), it may have either no solutions (infeasible solution and unbounded solution) or unreliable solutions (multiple solutions and degeneracy).

- 3.1 *Infeasible solutions* occur when no alternative satisfies all constraints. The problem may be over-constrained (too many constraints imposed). You are not given sufficient resources to use when producing a certain amount of output. Or you might mistakenly enter constraints (e.g., \leq instead of \geq) or incorrectly formulated the LP problem. If any nonzero artificial variable appears in the set of basic feasible variables in the final tableau of the simplex method, it indicates infeasible solution. See Albright & Winston (2005, pp.81-82).
- 3.2 *Unbounded solutions.* The optimal values of at least one of the decision variables are infinite (∞). Unbounded solutions will occur when necessary constraints are excluded from the LP formulation or constraints are incorrectly entered (e.g., \leq instead of \geq). We can recognize this solution when there is no positive coefficient in the pivot column in the tableau of any round except for the final tableau in the simplex method. In this case, the ranges of RHS and coefficients in the objective function cannot be calculated.
- 3.3 *Multiple solutions* occur when the objective function slope is the same as the slope of any constraint. There are many solutions that satisfy constraints and return the same optimal value of the objective function. You will observe multiple solutions if one or more of the allowable increases or decreases for the coefficients of the objective function are zero. In this circumstance, the slightest change in a coefficient would lead to a new corner and have the solution changed; yes, this solution is shaky.
- 3.4 *Degeneracy* (or degenerate solution) occurs if a corner which is defined by more than the minimum number of constraints also turns out to be the optimal corner. You can detect degeneracy, 1) if a constraint which is binding (zero slack or surplus) also has a zero

shadow price, 2) if the ranging information for some constraints has a zero allowable increase or decrease, and/or 3) if basic feasible variables have zero values in the final tableau in the simplex method. *Degeneracy is a problem particularly for shadow prices.* If we have too many binding constraints at the optimal corner, and we relax or tighten one of them, then the combination of constraints which are binding will change, and shadow price information is no longer reliable in case of degeneracy.

4. Sensitivity Analysis in the Objective Function

There are three groups of parameters in a LP problem: (1) coefficients in the objective function, (2) coefficients in LHS (constraints); and (3) RHS. Since coefficients in LHS are less likely to change in the short term, let us focus on (1) and (3) in this note.

The sensitivity analysis in the objective function focuses on coefficients of decision variables to examine their reduced costs and range information (allowable increase/decrease). Let us first examine how much a coefficient of the objective function can change (increase or decrease) without changing the current optimal solution ($x_1=2, x_2=6$).

Table 2. Sensitivity of Door Coefficient

Door	Window	Solution	Optimal Value
-\$100	\$500	(0, 6)	\$3,000
\$0	\$500	(2, 6)	\$3,000
\$100	\$500	(2, 6)	\$3,200
\$200	\$500	(2, 6)	\$3,400
\$300	\$500	(2, 6)	\$3,600
\$400	\$500	(2, 6)	\$3,800
\$500	\$500	(2, 6)	\$4,000
\$600	\$500	(2, 6)	\$4,200
\$700	\$500	(2, 6)	\$4,400
\$750	\$500	(4, 3) (2,6)	\$4,500
\$800	\$500	(4, 3)	\$4,700

Table 3. Sensitivity of Window Coefficient

Door	Window	Solution	Optimal Value
\$300	-\$100	(4, 0)	\$1,200
\$300	\$0	(4, 0)	\$1,200
\$300	\$100	(4, 3)	\$1,500
\$300	\$200	(4, 3) (2,6)	\$1,800
\$300	\$300	(2, 6)	\$2,400
\$300	\$400	(2, 6)	\$3,000
\$300	\$500	(2, 6)	\$3,600
\$300	\$600	(2, 6)	\$4,200
\$300	\$700	(2, 6)	\$4,800
\$300	\$750	(2, 6)	\$5,100
\$300	\$800	(2, 6)	\$5,400

Table 2 above illustrates what happened to optimal solution when changing the unit profit of a door, holding the unit profit of a window constant (comparative static analysis). When the unit profit decreases to below \$0 or increase to above \$750, the optimal solution (2, 6) will not be held any more. Table 3 suggests that the optimal solution will not change as long as the unit profit of window is greater than or equal to \$300; if the unit profit falls below \$300, the optimal solution also changes. Then, what do these observations mean?

4.1 Ranging information of the coefficient of a decision variable in the objective function tells us how flexible the objective function coefficient is or how much the coefficient of the decision variable can increase or decrease in the objective function while having the optimal solution remain unchanged. The optimal solution does not change as long as the unit profit of a door stays in \$0 through \$750 in Table 2 and \$200 through infinity in Table 3. As long as the coefficient of the decision variable changes within that range, the optimal solution remains unchanged but the value of the objective function will be changed accordingly due to the change in that coefficient.

4.2 The *allowable range of a coefficient* consists of its *lower bound* (= coefficient – allowable decrease) and its *upper bound* (=coefficient + allowable increase) and is expressed in the form of [lower bound, upper bound]. DO NOT let your boss calculate the allowable range

by “tossing” (presenting) allowable increase and decrease to him. Notice that an allowable increase and an allowable decrease are not necessarily symmetric.

Table 4. Excel Solver’s Sensitivity Report (Wyndor Glass)

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$C\$12	x1	2	0	300	450	300
\$D\$12	x2	6	0	500	1E+30	300

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$C\$14	Constraint 1 x1	2	0	4	1E+30	2
\$C\$15	Constraint 2 x1	12	150	12	6	6
\$C\$16	Constraint 3 x1	18	100	18	6	6

4.3 In Table 4, the current coefficient 300 of door (under “Objective Coefficient”) can change from \$0 (=300-300) to \$750 (=300+450). Read 450 under “Allowable Increase” and 300 under “Allowable Decrease.” The 0 is the lower bound of the door coefficient and 750 is its upper bound. The current optimal solution will remain unchanged as long as the coefficient of door stays within [\$0, \$750]. Double-check this allowable range from Table 2 and 4 above. Remember that the value of the objective function will change accordingly. Similarly, the coefficient of window can change from \$200 (=500-300) to infinity (=500+1E+30=500+10³⁰) without influencing the current optimal solution [\$200, Positive Infinity] (see 4.4 for details). Notice that the allowable increase of 1E+30 is 10³⁰ or the positive infinity (+∞). Check this allowable range from Table 3 and 4.

4.4 When a coefficient of an objective function changes to the maximum or minimum value of its allowable range, it might happen to observe changes in the optimal solution (of course, in the value of the objective function). When unit profit of a door becomes \$759 (upper bound) in Table 2, Excel Solver reports a new optimal solution of (4, 3). However, don’t be surprised at all! This solution is just one of multiple solutions since the slope of the objective function and that of constraint 3 becomes identical ($-\frac{3}{2} = -\frac{\$750}{\$500}$).

4.5 Not only (4, 3) but all points on the iso-profit line between (2, 6) and (4, 3) also return the same optimal value of the objective function. For example, one of multiple solutions (2, 6) brings us the same profit of \$4,500 (= \$700×2 + \$500×6). Similarly, the lower bound of \$200 for window in Table 3 results in the same slope $-\frac{3}{2} = -\frac{\$300}{\$200}$ and accordingly both (2, 6) and (4, 3) bring us the same profit of \$1,800 (= \$300×2 + \$200×6 = \$300×4 + \$200×3). Therefore, there is no actual change in the optimal solution as long as a coefficient stays within its allowable range. This issue is a matter of software (algorithm to handle multiple solutions) you use; other software might give you the same optimal solution (2, 6) under these two extreme cases.

5. Sensitivity Analysis in the Objective Function: Reduced Cost

5.1 *Reduced cost* asks how much the coefficient a decision variable in the objective function would decrease to start producing a particular output (more than or equal to 1 unit) in that

variable. The coefficient of the decision variable should decrease by that amount in order to have at least 1 as an optimal solution of that variable.

- 5.2 The reduced cost of a decision variable is NOT zero only when the decision variable has zero at the optimal solution. Conversely, if a decision variable has a positive value at the optimal solution, its reduced cost is always zero.
- 5.3 Read zero under “Reduced Cost” in *Variable Cells* in Table 4. Both reduced costs of x_1 and x_2 are zero since both decision variables have positive (nonzero) optimal values (values at the optimal solution).
- 5.4 In the Super Grain case (Table 5), the decision variable x_1 (the number of advertisement on TV spots) has a non-zero reduced cost of -50 since $x_1=0$ in the optimal solution. Variables x_2 and x_3 have zero reduced cost since they have non-zero values in the optimal solution (i.e., $x_2=20$ and $x_3=10$).

Table 5. Excel Solver’s Sensitivity Report (Super Grain)

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$C\$13	x1	0	-50	1,300	50	1E+30
\$D\$13	x2	20	0	600	150	50
\$E\$13	x3	10	0	500	300	33

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$C\$15	Constraint 1	4,000	3	4,000	1,000	1,500
\$C\$16	Constraint 2	1,000	5	1,000	600	200
\$C\$17	Constraint 3	0	0	5	1E+30	5

Maximize $\pi = 1300x_1 + 600x_2 + 500x_3$

subject to

$$300x_1 + 150x_2 + 100x_3 \leq 4000$$

$$90x_1 + 30x_2 + 40x_3 \leq 1000$$

$$x_1 \leq 5$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

- 5.5 From a theoretical view, Winston (2004) states, “The reduced cost for a nonbasic variable ... is the maximum amount by which the variable’s objective function coefficient can be increased before the current basis becomes suboptimal, and it becomes optimal for the nonbasic variable to enter the basis” (pp. 277-278). Hillier and Lieberman (2010) say, “[I]t is the minimum amount by which the unit cost of activity j would have to be reduced to make it worthwhile to undertake activity j (increase x_j from zero)” (p.234).
- 5.6 H&H (2011, CD-ROM) put, “The *negative* of the reduced cost indicates how much you would have to change the objective function coefficient of a zero-valued changing cell before it would become optimal for the changing cell to be nonzero (italics in original)” and conclude, “... the number of thousands of exposures per TV Spot would need to be at least 50 higher before it would be optimal to utilize TV spots.”

5.7 Alternatively, reduced cost is the shadow price of a non-negativity constraint. H&H (2011, CD-ROM) state, “The reduced cost for a zero-valued changing cell indicates the change in the objective function value per unit increase in that changing cell (after reoptimizing the other changing cells)” and “The reduced cost is the shadow price of the nonnegativity constraint for the corresponding changing cell. (This interpretation also holds for a changing cell with a nonzero value.)”

5.8 H&H (2011) interpret, “... after increasing the number of TV spots to 1, the new optimal solution becomes ... with a total of 16,950 ... (a decrease of 50).”; the reduced cost is “the rate at which the objective function value ... changes per unit increase in the nonnegativity constraint is -50 .”; and “[I]f we change the nonnegativity constraint from $\text{TV Spots} \geq 0$ to $\text{TV Spots} \geq 1$, then the objective function value (total number of thousands of exposures) will decrease by 50 to 16,950.” Table 6 reports the sensitivity analysis of Super Grain case with the new constraint $x_1 \geq 1$ added. Notice that the optimal solution changed $x_1=0 \rightarrow 1$, $x_2=20 \rightarrow 19$, and $x_3=10 \rightarrow 8.5$.

5.9 This 50 could be interpreted as *efficiency loss* of the additional constraint (forcefully assign non-negative value in the decision variable that would have zero in the optimal solution without the constraint)

Table 6. Excel Solver’s Sensitivity Report (Super Grain with Additional Constraint)

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$C\$13	X1	1	0	1,300	50	1E+30
\$D\$13	X2	19	0	600	150	50
\$E\$13	X3	8.5	0	500	300	33

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$C\$15	Constraint 1	4,000	3	4,000	850	1,425
\$C\$16	Constraint 2	1,000	5	1,000	570	170
\$C\$17	Constraint 3	1	0	5	1E+30	4
\$C\$18	Constraint 4	1	-50	1	4	1

Maximize $\pi = 1300x_1 + 600x_2 + 500x_3$

subject to

$$300x_1 + 150x_2 + 100x_3 \leq 4000$$

$$90x_1 + 30x_2 + 40x_3 \leq 1000$$

$$x_1 \leq 5$$

$$x_1 \geq 1$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

5.10 If Super Grain wants to (forcefully) advertise at least once on Saturday morning TV programs for children, the coefficient of x_1 (expected number of exposures in thousand) in the objective function should be reduced at least by -50 (or increase by 50) or change from the current 1,300 to 1,350. In other words, when the coefficient of x_1 increases at least by 50 (or more than 50), you will find (by rerunning LP) that the new optimal solution will have more than or equal to 1 unit of TV advertisement; of course, the value of objective function (outcome) will be changed accordingly. For an additional TV advertisement, the expected number of exposures will decline by 50 thousands as long as the

number of TV advertisement stays $[0, 5]$. See Albright & Winston (2005, pp.70-74) and Hillier & Lieberman (2010, pp.233-235).

6. Sensitivity Analysis in Constraints

Suppose a minister observes a long delay in a public project and asks Congress to give more money or appropriate additional budget for the project, assuming the shortage of money is the main reason for the delay. Then Congress will ask, “Is this additional money worth considering?” and “If yes, how much is the appropriation worthwhile?” He or she must provide evidence that money is the key factor to make progress in the project. He also must show the extent to which additional money makes difference (marginal amount of progress in the project).

For instance, 1) money is the most binding constraint of the project and 2) the project will progress by .05 percent for every 1 million dollars. If Congress give 100 millions, we can expect 5 percent additional progress ($5.0\% = .05\% \times 100$ since we assume proportionality in LP). Then, the Congress will decide whether or not to provide 100 millions on the basis of this information. If the answer for the question 1 is “No,” then decline the request right away. If “Yes” in question 1, but the marginal amount (.05%) is not substantial (smaller than Congress’ expectation, say .1%), then Congress’ answer will be “No.”

Table 7. Sensitivity of the Shadow Price of Constraint 1 (Plant 1)

LHS	RHS	Slack/ Surplus	Shadow Price	Allowable Increase	Allowable Decrease	Allowable Range	Optimal Solution	Optimal Value
0	0	0	\$300	2	0	$[0, 2]$	(0, 6)	\$3,000
1	1	0	\$300	1	1	$[0, 2]$	(1, 6)	\$3,300
2	2	0	0	1E+30	0	$[2, \infty]$	(2, 6)	\$3,600
2	3	1	0	1E+30	1	$[2, \infty]$	(2, 6)	\$3,600
2	4	2	0	1E+30	2	$[2, \infty]$	(2, 6)	\$3,600
2	5	3	0	1E+30	3	$[2, \infty]$	(2, 6)	\$3,600
2	6	4	0	1E+30	4	$[2, \infty]$	(2, 6)	\$3,600
2	0	1E+30	...	$[2, \infty]$	(2, 6)	\$3,600

Let us first change the RHS of constraint 1 (plant 1) somehow and then see what will happen in the objective function. Table 7 shows that the optimal values of the objective function remain unchanged until the RHS decreases to 2 (2 units decrease from the current 4). If the RHS declines to 1, the optimal solution will be (1, 6) and the value of objective function will be \$3,300 ($=\$300 \times 1 + \500×6). An increase in the RHS beyond 12 does not seem to influence the optimal value of the objective function. A manager is not interested in this constraint since any change in this constraint does not make big difference in the profit.

Now, investigate how change in the RHS of constraint 2 (plant 2) influences the optimal value (Table 8). When the RHS becomes 11 (one unit decrease from the current 12), the optimal solution will change to (2.3, 5.5) and accordingly the value of objective function will be \$3,450 ($=\$300 \times 2.3 + \500×5.5), which is \$150 smaller ($=\$3,600 - \$3,450$) than the current \$3,600. This \$150 is a marginal change in profit for every unit change in the RHS. When the RHS decreases to 6, the optimal solution will be (4, 3) and the optimal value is \$2,700. The decrease in profit is \$900 ($=\$3,600 - \$2,700$) and is alternatively calculated as $\$150 \times 6$. If we decrease the RHS of plant 2 down to 5, the optimal value will be \$2,450 ($=\$300 \times 4 + \500×2.5) at the optimal solution of (4, 2.5). Notice that the marginal change in the profit is not \$150 any more but becomes \$250 ($=\$2,700 - \$2,450$).

Table 8. Sensitivity of the Shadow Price of Constraint 2 (Plant 2)

LHS	RHS	Slack/ Surplus	Shadow Price	Allowable Increase	Allowable Decrease	Allowable Range	Optimal Solution	Optimal Value
5	5	0	\$250	5	1	[4, 10]	(4, 2.5)	\$2,450
6	6	0	\$150	12	0	[6, 18]	(4, 3)	\$2,700
7	7	0	\$150	11	1	[6, 18]	(3.7, 3.5)	\$2,850
...	...	0	\$150	[6, 18]
11	11	0	\$150	7	5	[6, 18]	(2.3, 5.5)	\$3,450
12	12	0	\$150	6	6	[6, 18]	(2, 6)	\$3,600
13	13	0	\$150	5	7	[6, 18]	(1.7, 6.5)	\$3,750
...	...	0	\$150	[6, 18]
17	17	0	\$150	1	11	[6, 18]	(.3, 8.5)	\$4,350
18	18	0	\$150	0	12	[6, 18]	(0, 9)	\$4,500
18	19	1	0	1E+30	1	[18, ∞]	(0, 9)	\$4,500

When increasing the RHS of plant 1 from 12 to 18, the optimal solution and optimal value change accordingly with exactly the same marginal increase in the profit. However, if the RHS becomes larger than 18, the marginal increase becomes zero, not \$150. When comparing constraints 1 and 2, a manager can realize that constraint 2 (available operation hours per week in plant 2) is a critical factor that influences the profit of the company and will decide a right level of investment to increase operation hours of plant 2. The \$150 here is called shadow price, which provide important information for managerial decision-making.

7. Sensitivity Analysis in Constraints: Shadow Price

7.1 *Shadow price* (or *dual value*) asks how much better the objective function would be if the RHS of a constraint could be changed a little bit. This shadow price contains very important implications for decision makers. Shadow price tells us something about the maximum we should be willing to pay to relax a constraint (or to increase the value of RHS) by one unit. Relaxing a constraint means that the feasible set becomes expanded and more alternatives will be considered. See Albright & Winston (2005, pp.70-74) and Hillier & Lieberman (2010, pp.131-136).

7.2 Shadow price is not an actual (market) price of a product, of course. Shadow price is the amount of improvement in the objective function that occurs when we relax a constraint by one unit (add one unit of available resource to the constraint). That is, for each additional unit of a resource (RHS value of the constraint), the amount of value (profit or cost) of the objective function is expected to increase by that shadow price. The beauty of shadow price is that you can predict the increase in the objective function WITHOUT rerunning LP when changing the RHS value of a constraint. That is, we can predict the change in the optimal value when changing the RHS values of constraints 1 and 2 without Table 5 and 6.

7.3 In Table 4, find shadow prices of constraints 1 through 3 under “Shadow Price” of the *Constraints* section; they are 0, 150, and 100, respectively. The marginal increase was \$0 in Table 7 and \$150 in Table 8. The values of RHS (maximum capacity or resources available) are listed under “Constraint R.H.Side” and those resources consumed at the optimal solution under “Final Value.” Their difference is the slack of a constraint. For instance, the slack of the constraint 1 (plant 1) is 2 (= 4 - 2).

7.4 *Complementary slackness* says that if a constraint is binding and has zero slack/surplus, its shadow price is not zero. Conversely, if the shadow price of a constraint is not zero, then the constraint must be binding and have zero slack or surplus. The shadow price of a nonbinding constraint is zero. In Table 1 and 4, check that nonbinding constraint 1 has a nonzero slack (2) and zero shadow price. And make sure that binding constraints 2 and 3 have zero slack and nonzero shadow prices of \$150 and \$100, respectively. A violation of this complementary slackness indicates a serious problem (e.g., degeneracy) in the optimal solution and shadow price.

7.5 *Interpreting the value of shadow price*. Whenever the firm increases the maximum number of production hours per week (capacity or available resources) of plant 2 by 1 unit, the profit will increase by \$150. Yes, the optimal solution will be changed accordingly. For example, if the firm increases the maximum capacity from current 12 hours to 15 hours (3 unit increase), the profit will increase from current \$3,600 to \$4,050 ($=3,600+150\times 3$). Similarly, for every 1 unit increase in the maximum resources of plant 3 from the current 18 hours, Wyndor Glass Co. will get \$100 increase in its profit. The zero shadow price of constraint 1 means that there is no increase in the profit when the firm increases the value of RHS (available resources) of plant 1 from the current level of 4 hours to 5 hours.

7.6 Therefore, policy analyst and public managers should pay attention to plant 2 and 3 because their capacities (available resources) are key factors to maximize the profit. They might be interested in investing more money to increase resources available in plant 2 and/or 3. If the unit cost of increasing production hours is the same in both plants and Wyndor Glass has to choose only one plant, the company will pour money on plant 2 to take advantage of \$150 over \$100.

8. Sensitivity Analysis in Constraints: Allowable Range of a Value of RHS

8.1 *Ranging information of the value of a RHS* tells you to what extent a value of RHS of a constraint can change without changing the value of the corresponding shadow price. In other words, the shadow price remains valid as long as the value for RHS of a constraint stays within the allowable range, $RHS \pm$ (allowable increase/decrease).

8.2 In Table 4, find an allowable increase of 6 and an allowable decrease of 6 for constraint 2 and 3. We happened to have the same allowable increase and decrease in this case. We may ignore the allowable increase and decrease of the constraint 1. Remember that this constraint is not binding and has zero shadow price (complimentary slackness).

8.3 The RHS range of constraint 2 is $[12-6, 12+6]=[6, 18]$. Check this range in Table 8. When the maximum resource of plant 2 changes but stays within $[6, 18]$, the current shadow price of \$150 remains valid although the optimal solution and optimal value will change somehow.

8.4 Conversely, if the value of RHS goes beyond that allowable range (e.g., decreases down to 3 or increases up to 20), the shadow price of \$150 may not be applied. You don't know what the value of the objective function will be in that circumstance. Hence, you MUST rerun the LP to obtain new shadow prices and new optimal solution.

8.5 The lower and upper bounds of RHS of constraint 3 are 12 (=18-6) and 24 (=18+6), respectively, forming the allowable range [12, 24] for plant 3.

Table 9. Sensitivity Analysis for Objective Function Coefficients and RHS Values

	Coefficient of a Decision Variable	Value of Right-hand Side (RHS)
Sensitivity	Reduced cost	Shadow price
Condition for being nonzero	The value of a decision variable is zero in the optimal solution	The slack/surplus of a constraint is zero (binding constraint) in the optimal solution
Change, where?	Objective function (coefficients)	Constraints (RHS)
Interpretation (meaning)	Minimum amount of reduction of a coefficient to force its decision variable to be produced at least 1 unit or the shadow price of a non-negativity constraint.	Marginal change in the value of the objective function for unit increase in the RHS value of a constraint
Impact on the optimal solution	Will change if the coefficient reduces at least by that minimum amount.	Will change when the RHS value increases or decreases within the allowable range.
Impact on the objective function	Will change because of new coefficients in the objective function.	Will change because of the new RHS value and new optimal solution.
What if zero?	No impact on the optimal solution but a new objective function as long as the coefficient stays within the allowable range.	No impact on the optimal solution and the value of objective function as long as the RHS value stays within the allowable range.
Caution		Not reliable in case of degeneracy
Allowable range	Tolerable upper and lower bounds (range) of the coefficient of a decision variable without changing the optimal solution. Current coefficient \pm allowable increase/decrease.	Tolerable upper and lower bounds (range) of a RHS value that hold the shadow price valid. Current RHS \pm allowable increase/decrease, [RHS-allowable decrease, RHS+allowable increase]
Allowable increase/decrease	Amount of increase or decrease that the coefficient can change without changing the optimal solution.	Amount of increase or decrease that the RHS value can change while holding the shadow price valid.
What if out of bounds?	If a coefficient changes out of its allowable range, the optimal solution will change; you need to run LP with the new coefficient to get the new optimal solution.	If the value of a RHS changes out of its allowable range, the shadow price is no longer valid; you need to run LP with new RHS to get new shadow prices.
Symmetric?	Allowable increase and decrease are NOT necessarily symmetric.	
Software issue	SAS/OR reports the upper and lower bounds (allowable range), while Excel Solver returns allowable increase/decrease (\pm change) only	

* All analyses are done under the *ceteris paribus* assumption (“holding all others things constant”).

9. Ceteris Paribus Assumption in Sensitivity Analysis

Sensitivity analysis (i.e., reduced cost and shadow price) basically assumes that only one of the RHS values (or the coefficient of a decision variable in the objective function) is changing at a time, “holding all other variables being equal or constant.” This *Ceteris Paribus* assumption is to rule out compounding effects that will occur when more than one value changes at a time. *Ceteris paribus* is common in a comparative static analysis. Recent trends ask, however, “What will happen in the optimal solution and value of objective function if more than one parameter (e.g., coefficients of the objective function and RHS values of constraints) changes simultaneously?” Thank to progress in technology, it is not painful nowadays to run LP to get such sensitivity information. For effects of simultaneous changes, see 5.4 and 5.6 of H&H (2014).

End of this document.