

INTERNATIONAL UNIVERSITY OF JAPAN
Public Management and Policy Analysis Program
Graduate School of International Relations

DCC5350 (2 Credits)
Public Policy Modeling
Spring 2015

Illustration of a Simple Linear Programming

This class note illustrates how a simple LP problem can be formulated and interpreted. The problem here is to minimize the cost of purchasing apples and pears while satisfying all constraints. This is an example of a typical blending problem.

1. Steps in LP Formulation

1.1 Problem and relevance of LP formulation: The question here is to decide optimal numbers of apples and pears to be purchased at AEON. This simple question can be properly answered by LP. But this LP formulation is only for a pedagogical purpose.

1.2 Classification of this problem: programmed (purchasing fruits is so common), certainty with some exceptions (not being able to know quality of fruits correctly), static (considering this time of purchase only), and non-strategic (although AEON will change their long-term supply strategies depending on customers' demand for apple and pear).

1.3 Decision variables: you can decide the numbers of apples and pears to be purchased. You cannot change the price of apples or pears. Let x_1 and x_2 respectively denote the number of apples and pears to be purchased.

1.4 Environment variables: Weather, market condition, regulations on farming, etc.

1.5 Criterion variable (objective function): You need to evaluate this purchasing problem and may say, "to minimize the cost of purchasing apples and pears." The unit prices of apples and pears are ¥200 and ¥300, respectively. Its corresponding mathematical expression will be, "Min $Z=200x_1 + 300x_2$."

1.6 You are encountering three constraints. The first constraint states, "The number of apples purchased should be at least 3 (or greater than or equal to 3)." Its corresponding mathematical expression is, " $x_1 \geq 3$." The second constraint is "The number of pears purchased should be at least 4" and its corresponding expression is " $x_2 \geq 4$." Finally, "The total number of fruits you purchased should be at least 10" can be expressed as, " $x_1 + x_2 \geq 10$."

1.7 Yes, you need to add nonnegativity because you cannot purchase, say, -5 apples; that is, " $x_1 \geq 0$ and $x_2 \geq 0$."

1.8 Check unit of measurement and scale. First check the objective function. 200 is measured in ¥/apple or ¥/EA and 300 in ¥/pear or ¥/EA. x_1 and x_2 are number of apples and pears. Therefore, $Z = 200x_1 + 300x_2 = (\text{¥/apple}) \times \text{apple} + (\text{¥/pear}) \times \text{pear} = \text{¥}$. Since Z is the cost (or expense), it makes quite sense. Then check constraint 1. " $x_1 \geq 3$ " is "apple \geq apple" or "EA \geq

EA.” Yes, this is commensurable. Constraint 2 “ $x_2 \geq 4$ ” is also commensurable since “pear \geq pear” or “EA \geq EA.” Finally, “ $x_1 + x_2 \geq 10$ ” is “apple + pear \geq apple or pear” or “EA + EA \geq EA.” Remember that you can purchase at least 10 apples and pears. Of course, the unit of measurement for constraint 3 is okay. The scale is not an issue because there is no too big and too small numbers.

2. LP Formulation

Now, the LP formulation is as follows. Do not add any unnecessary information such as, “This is the objective function,” “This constraint says, the number of ...,” and “nonnegativity is also needed.” Please take advantage of conciseness and simplicity of mathematics. Do not say, “Constraints are:” or “Conditions:” instead of “Subject to” or “S.T.” Use proper jargons and be professional.

Min $Z = 200x_1 + 300x_2$
 Subject to:
 $x_1 \geq 3$
 $x_2 \geq 4$
 $x_1 + x_2 \geq 10$
 $x_1 \geq 0, x_2 \geq 0$

	A	B	C	D	E	F
1	Purchasing Apples and Pears (Advanced)					
2						
3	Summarize: minimize the cost of purchasing apples and pears given constraints.					
4						
5	Raw Data	x1	x2		RHS	
6	Apple minimum	1	0			3
7	Pear minimum	0	1			4
8	Total minimum	1	1			10
9	Price	\$200	\$300			
10						
11	Coefficients	\$200	\$300			
12	Decision	6	4			
13						
14	Objective			\$2,400	=200*x1 + 300*x2	
15						
16	Constraints	LHS			RHS	
17		6		>=		3
18		4		>=		4
19		10		>=		10
20						
21	Nonnegativity					

3. Translation of LP Formulation into Excel Solver

Now you need to translate the LP formulation into Excel Solver. Provide raw data and proper mathematical expression as follows. See the Excel file (purchasing_fruits.xls on the course Web page). Decision variables are located in B12 and C12. You may leave empty for these two cells; Excel Solver will fill some values in them for you.

The objective function is in D14 where you have to provide mathematical expressions using cell references. For example,

$$200*B12 + 300*C12$$

You should not type in, “200*x1+300*x2” in D14 (Excel will refer X1 and X2 cells instead of B12 and C12). Since you have price information in B11 (=200) and C11 (300), you can improve the expression using these cells.

$$B11*B12 + B11*B12$$

But more advanced one is to use SUMPRODUCT() as follows.

SUMPRODUCT(B11:C11, B12:C12)

SUMPRODUCT() allows you to minimize silly mistakes and make you LP formulation more systematic. This function is very powerful especially when you have many decision variables and constraints. Caution! Do not tell Excel like, SUMPRODUCT(b11:b12, c11:c12), since it means $b11*c11+b12*c12$, NOT $B11*B12 + C11*C12$.

And you need to provide mathematical expressions for the LHS of three constraints.

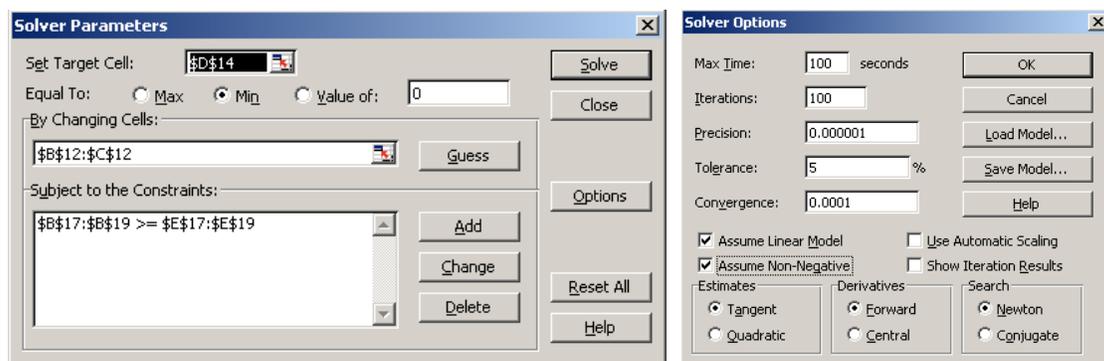
SUMPRODUCT(B6:C6, B12:C12) in B17

SUMPRODUCT(B7:C7, B12:C12) in B18

SUMPRODUCT(B8:C8, B12:C12) in B19

The RHS value of constraint 1 is 3 or E6. RHS of constraint 2 and 3 are E7 (=4) and E8 (=10), respectively.

In Excel Solver, you need to provide following cell references in proper places. You must make sure the type of optimization (minimization in this case).



Click on Options and check “Assume Linear Model” and “Assume Non-Negative” and click OK. Finally, click on Solve to solve the LP problem.

4. Interpretation of Optimal Solution and Objective Function

Once you formulate this LP problem using Excel correctly, you may get the following Answer Report.

From the Table labeled “Target Cell (Min),” you should learn that this LP formation is a minimization and the mathematical expression of the objective function is in D14. The optimal value is ¥2,400. (Treat \$ as if ¥)

The Table with a heading “Adjustable Cells” reports the optimal solution: $x_1 = 6$ and $x_2 = 4$. That is, when you purchase 6 apples and 4 pears, you can satisfy all constraints and minimize the cost of purchasing fruits. The minimum cost is calculated as $¥200 \times 6 + ¥300 \times 4 = ¥2,400$.

Target Cell (Min)

Cell	Name	Original Value	Final Value
\$D\$14	Objective	\$0	\$2,400

Adjustable Cells

Cell	Name	Original Value	Final Value
\$B\$12	Decision x1	0	6
\$C\$12	Decision x2	0	4

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
\$B\$17	LHS	6	\$B\$17>=\$E\$17	Not Binding	3
\$B\$18	LHS	4	\$B\$18>=\$E\$18	Binding	0
\$B\$19	LHS	10	\$B\$19>=\$E\$19	Binding	0

5. Interpretation of Slack and Surplus

The third table with the “Constraints” heading provides RHS values and slack/surplus information of constraints. The column “Cell” indicates the locations of the LHS of constraints. For example, the LHS expression equivalent to “ $x_1 + x_2 \geq 10$ ” is stored in B19 in the Excel worksheet. “Cell Value” contains the values of LHS at the optimal solution. The LHS values of constraint 1 and 2 are 6 and 4. The LHS value of constraint 3 is 10, which is calculated as $x_1 + x_2 = 6 + 4$.

Since “Formula” contains \geq , the constraints have surplus (not slack). Surplus, LHS-RHS, is calculated as follows.

$$3 = 6 - 3$$

$$0 = 4 - 4$$

$$0 = 10 - 10$$

Note that the RHS values are reported in Sensitivity Report in Excel. By definition, if a constraint has a zero surplus (or slack), it is binding; if a constraint has nonzero surplus, it is not binding. Therefore, constraint 1 is not binding because of a nonzero surplus 3, while constraints 2 and 3 are binding since their surpluses are zero. Excel reports slack or surplus in the “Slack” column and binding constraints under “Status.”

6. Interpretation of Sensitivity Analysis

Sensitivity Report provides sensitivity analyses on the objective function and constraints.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$12	Decision x1	6	0	200	100	200
\$C\$12	Decision x2	4	0	300	1E+30	100

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$B\$17	LHS	6	0	3	3	1E+30
\$B\$18	LHS	4	100	4	3	4
\$B\$19	LHS	10	200	10	1E+30	3

6.1 Reduced Costs and Objective Coefficients

From the table labeled as “Adjustable Cells,” you should find the optimal solution, $x_1 = 6$ and $x_2 = 4$, in the “Final Value” column, and coefficients of the objective function under “Objective Coefficient” (see 200 and 300). You should find zero reduced cost for two objective coefficients, indicating two decision variables have positive values at the optimum. The “Allowable Increase” and “Allowable Decrease” columns provide the extent that each coefficient can change although this range information is meaningless in case of zero reduce cost. For instance, the coefficient 200 can increase by 100 and decrease by 200. Therefore, its lower and upper bounds are 0 ($=200-200$) and 300 ($=200+100$), respectively.

6.2 Shadow Prices and RHS Values

The table with the “Constraints” heading provide shadow prices of individual constraints. The numbers under “Final Value” are the (calculated) LHS values of constraints whereas the numbers under “Constraint R.H. Side” are the RHS values of corresponding constraints. For instance, constraint 3 “ $x_1 + x_2 \geq 10$ ” is “ $6 + 4 \geq 10$ ” at the optimal solution and the LHS is calculated as 10 ($= 6 + 4$) and RHS is 10. Constraint 1 has zero shadow price since it has surplus 3 ($=6-3$). By contrast, constraint 2 and 3 have nonzero shadow prices since they are binding.

The shadow price 100 of constraint 2 is interpreted as, “If you increase the RHS value of constraint 2 by 1 (from 4 to 5), the value of the objective function will increase by 100.” Alternatively, “per one unit increase in the minimum number of apples purchased, you need to pay ¥100 more.” This interpretation is valid until the increase and decrease of RHS do not exceed limits. The “Allowable Increase” and “Allowable Decrease” tell you how much the RHS value can increase or decrease without changing the meaning of the shadow price. If the current RHS value of constraint 2 decreases by down to 4 or increases by up to 3 or if the RHS value stays within 0 ($=4-4$) and 7 ($=4+3$), the shadow price 100 does not change. Otherwise, you do not know the correct shadow price unless you solve the LP problem with new RHS value again. The 0 and 7 are respectively called lower and upper bounds of the RHS value of constraint 2. Excel Solver does not provide lower and upper bounds and reports allowable decrease and increase; but you must provide lower and upper bounds to your boss (Do not let your boss calculate lower and upper bounds!).

End of this document.