INTERNATIONAL UNIVERSITY OF JAPAN
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This note is based on David Good's lecture, Hillier and Hillier (2014), Albright and Winston (2005), and Hillier and Lieberman (2010). This note SHOULD NOT be used as a substitute of the textbook.

Linear Programming Formulation

Linear Programming (LP) is a representative type of *mathematical programming* (or mathematical optimization) and was pioneered by George B. Dantzig in the U.S. Air Force, who developed the simplex method in 1947 to solve military logistics problems.[1] LP has been widely used in the private sector (e.g., manufacture and aviation business) but less frequently employed in the public sector largely due to its somewhat strong assumptions. But its model formulation and core concepts (e.g., reduced cost and shadow price) are still valuable to policy analysts and public managers.

1. Underlying Assumptions

LP has core underlying assumptions of certainty, divisibility, and linearity (proportionality, additivity, and homogeneity), which are seldom satisfied precisely in the real world. See Albright & Winston (2005: 78-81) and Hillier & Lieberman (2010: 36-42).

1.1 *Certainty* (deterministic or nonstochastic) Every piece of information (i.e., coefficients in the objective function and constraints) is assumed to be well known. LP is deterministic without any risk and uncertainty. The environment variables (the right-hand side values of constraints) are NOT probabilistic. Outcomes do not follow any probability distribution. If coefficients (e.g., productivity and market price) are affected by weather (e.g., rain, temperature, humidity, and wind) and social events (e.g., war, strike, recession, and disease), this assumption is violated. Under this certainty, LP often uses "average" to summarize the reality.

1.2 *Divisibility* The legitimate values of decision variables are <u>non-negative real numbers</u> or whole numbers. Variants of LP such as integer programming and binary programming can handle this indivisiblility of integer and binary decision variables.

1.3 *Linearity* includes proportionality, additivity, and homogeneity that are closely related to each other.
- *Proportionality* means that outcome variables are proportional to the level of decision variables. That is, coefficients in constraints or the objective function are constants. If the price of a product is $100.00, the amount of earnings is proportional to sales of the product (the number of products sold): earnings=100 × sales. Diminishing return (law

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[1] The original name is *Programming of Interdependent Activities in a Linear Structure*.

of increasing relative cost or opportunity cost) is an example of the violation of this proportionality.

- *Additivity* means that the production possibility frontier is a straight line. The contribution of a decision variable to the objective and any constraint is independent of other decision variables. The objective function and constraints are expressed by linear combination of decision variables. This assumption rules out synergies or complementarities between decision variables (no interaction term is allowed in the objective function and constraints).
- *Homogeneity* requires that all units of decision, outcome, and environment variables are assumed to be identical. For example, individual employees have the same productivity; materials (e.g., gas, water, and timber) have the same quality; and each type of machines (e.g., snow blower and excavator) has the same function and efficiency; evacuator 1 and 2 are assumed to be equivalent even if their ages might be different and different drivers with different levels of skills operate them.

2. Components of Mathematical Programming

LP consists of decision variables, objective function, constraints, and nonnegativity.

2.1 Decision variables are alternatives that are settable (controllable) by decision makers. These variables should appear in the "By Changing Cells" box of Excel Solver.

2.2 Objective function is a mathematical expression for a single goal (e.g., maximizing profit or minimizing cost) that you want to achieve: $Max(\Sigma\beta_i x_i)$ or $Min(\Sigma\beta_i x_i)$ where β_i is the coefficient of a decision variable x_i. Keep in mind that "$\Sigma\beta_i x_i$" alone cannot be a relevant objective function. This objective function should appear in the *Set Target Cell*" (in an old version) or the *Set Objective* input box in Excel Solver. And then choose the type of optimization (i.e., maximization or minimization).
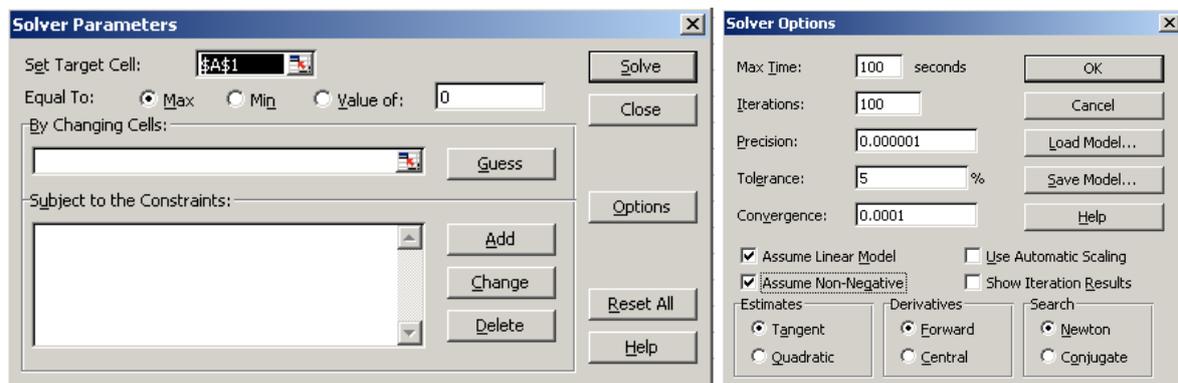
2.3 Constraints describe environmental, technological, legal, and other factors that limit your choices. A constraint compares an outcome variable that is determined by a combination of decision variables on the left-hand side (LHS) and its associated environment variable on the right-hand side (RHS). There are three forms of constraints.

- $\Sigma\delta_i x_i \leq c_i$ (The outcome should not exceed the maximum limit set by a constraint)
- $\Sigma\delta_i x_i \geq c_i$ (The outcome should not smaller than the minimum limit set by a constraint)
- $\Sigma\delta_i x_i = c_i$ (The outcome should be equal to the value set by a constraint)

2.4 In order to impose a constraint, try to find out a relationship between a linear combination of decision variables on the left-hand side and its RHS value. Simplify the expression of a

relationship and then arrange the linear function logically so that only RHS value appears on the right-hand side.

2.5 Constraints should appear in the *Subject to the Constraints* input box in Excel Solver. Take a look at the following expressions carefully.

- $2x_1 - 3x_2$ <= 10 (O) $- 15x_2 + 10x_1$ <= 50 (X)
- x_1 $- 4x_3$ >= 5 (O) $- 4x_3$ <= 5 - x_1 (X)

2.6 *Nonnegativity* restricts decision variables to having nonnegative values: $x_i \geq 0$. In general, it is not possible that a decision variable has a negative value (e.g., the number of snow blowers mobilized is -10. Does this sentence make sense?). Excel Solver has an option of *Assume Non-Negative* (in an old version) or *Make Unconstrained Variables Non-negative*.

3. Types of Linear Programming

Product mix and blending problems are most common in LP.

3.1 *Product mix problem* maximizes the value (profit) of the output produced given a set of resources available. The objective function is maximized while satisfying constraints for limited recourses ($\leq$ type constraints). For instance, "How do I have to use sprinkler systems and snow blowers to maximize the amount of snow removed per day?"

3.2 *Blending problem* minimizes the cost of production given a set of output requirements determined by decision makers. For example, "How do I have to use sprinkler systems and snow blowers to minimize the cost of removing 9,000 tons per day?"

3.3 In a *transportation problem*, decision variables involve matching origins and destinations and the objective function is often to minimize travel time or cost subject to supply constraints (resource limitations or capacities) and demand constraints (minimum requirement to be transported).

3.4 *Assignment problems* always have exactly one unit for each demand (task) and exactly one unit for each source supply (person). Their solutions are integer (integer programming)

3.5 Other variants of LP
- Integer programming
- Binary programming to handle binary decision variables
- Goal programming

4. Formulating a LP Problem

4.1 Evaluate the relevance of applying LP to the problem in question. Does this problem fit a standard LP pattern (product mix or blending)? You need to check if all pieces of necessary information (e.g., market prices, labor, capital, equipment, and environment) are available with certainty. And then check if LP's underlying assumptions can be justified. Wyndor Glass Co. has a classical product mix problem to look for the optimal solution that can maximize the value of objective function.

4.2 Determine *decision variables* and assign relevant symbols (e.g., x_1, x_2, x_3… rather than x, yy, xyz). Ask yourself if you can control the values of policy variables or if they are determined by others (e.g., your president or environment). You must include the proper *units* in which the variables are measured. If you define a variable and its unit badly, the objective function and constraints oftentimes do not make sense. In the example, decision variables are defined as D (the number doors to produce per week) and W (the number of windows to produce per week) rather than x_1 and x_2 for the pedagogical purpose. Their units are EA (each).

4.3 Define a *criterion variable* using a mathematical (linear) expression of decision variables (policy variables). This is the *objective function*. You may maximize the profit (product mix) or minimize the cost of production (blending). In the example, the question is how many doors and windows Wyndor Glass Co. should produce to maximize its profit. Therefore, the objective function is to maximize the profit by seeking the best combination (optimal solution) of the numbers of doors and windows to produce.

Maximize *P* = 300D + 500W

4.4 Incorporate constraints and compare outcome variables expressed in a combination of decision variables (left-hand side) and its corresponding environment variables (right-hand side). This example has three conditions (constraints). "A door needs one hour for producing a aluminum frame in plant 1 that has maximum capacity of 4 hours per week" (words): D ≤ 4 (mathematical expression); "A window needs two hours for producing a wood frame in plant 2 with maximum capacity of 12 hours per week": 2W ≤ 12. Finally, "a door needs three hours and a window needs two hours for assembling in plant 3 whose maximum capacity is 18 hours per week": 3D + 2W ≤ 18.

subject to
$$D \qquad \leq 4$$
$$2W \leq 12$$
$$3D + 2W \leq 18$$

4.5 Include nonnegativity constraints. Both D and W must not be negative; you are never, ever able to produce -5 doors!

D ≥ 0, W ≥ 0

4.6 Check the unit of measurement so that a combination of coefficients and decision variables in the objective function and constraints are *commensurable* with the criterion variable (e.g., profit or cost) and right-hand side value (environment variable), respectively. For example, D and W respectively have their units of EA/week (the number of doors and windows to produce per week); their coefficients are measured in the unit price ($/EA); and the result of the objective function is $ (amount of earnings per week in USD). Accordingly, the objective function Max P = $300D+$500W has reasonable units: $/Week = ($/~~EA~~)×(~~EA~~ /Week) + ($/~~EA~~)×(~~EA~~ /Week). What if the unit price of a window is measured in Yen (=¥50,000)? Then, you are just off the tract! ($/EA)×(EA/Week) + (¥/EA)×(EA/Week) = ??? And what if one decision variable is measured in Celsius and its coefficient in Fahrenheit? Completely absurd! Therefore, you must check unit and, if necessary, standardize different units (e.g., either $ or ¥ in the above example). Take a look at the following example.

Objective function: Max P = 300D + 500W (Math expression)
$/Week = ($/~~EA~~)×(~~EA~~/Week) + ($/~~EA~~)×(~~EA~~/Week)

Plant 1:   1D ≤ 4 (Math expression)
                     (Hours/~~EA~~)×(~~EA~~/Week) ≤ Hours/Week

Plant 2:   2W ≤ 12 (Math expression)
                     (Hours/~~EA~~)×(~~EA~~/Week) ≤ Hours/Week

Plant 3:   3D + 2W ≤ 18 (Math expression)
                     (Hours/~~EA~~)×(~~EA~~/Week) + (Hours/~~EA~~)×(~~EA~~/Week) ≤ Hours/Week

4.7 Check if *scales* (a range of numbers used to measure a variable) are reasonable; otherwise, adjust them in order to <u>avoid too big or too small numbers</u>. A too big number may result in an "overflow" problem, whereas a too small number ends up getting misleading numbers due to "underflow." What if the unit price of a window is measured in trillion dollar (i.e., too small number of .000,000,000,005) or Rupia (i.e., too big number of 5,000,000,000)? Fortunately, this LP example does not have such scale issue that is discussed in Albright & Winston (2005: 80-81).

4.8 Translate the objective function (including decision variables) and constraints into the expressions that computer software packages (e.g., SAS/OR, LINDO, Corel Quattro Pro Optimizer, and Excel Solver) can understand. For data input in Excel, see 2.2 and 2.5 of H&H (2014) and Albright & Winston (2005: 64-66). The following script is an example of solving the Wyndor Glass's problem in SAS/OR, which assumes nonnegativity internally.

```
DATA wyndor_glass;
INPUT _ID_ $ x1 x2 _TYPE_ $ _RHS_;
DATALINES;
object 300 500 MAX .
plant1  1   0 LE 4
plant2  0   2 LE 12
plant3  3   2 LE 18
RUN;

PROC LP; RUN;
```

4.9 Once you get the result, interpret and evaluate it to see if the optimal solution makes sense and how reliable the solution is (sensitivity analysis). Then present the solution and policy suggestions to your clients (e.g., your boss or Congressman).

**5. Graphical Method to Solve a LP Problem**

Once you formulate a LP problem, you may solve LP problems using (1) a graphical method, (2) simplex method, or (3) computer software packages like SAS/OR and Excel Solver. A graphical method identifies the feasible set defined by constraints; finds the corners of the feasible set; evaluates the value of the objective function—*iso-profit* line— at each corner; and then determines the corner that has the best value. This method is quite intuitive but limited by the number of decision variables; more than three variables cannot be visualized.

5.1 The product mix problem of producing doors and windows is summarized as:

Maximize $P = 300D + 500W$
subject to
$$D \leq 4$$
$$2W \leq 12$$
$$3D + 2W \leq 18$$
$$D \geq 0, W \geq 0$$

5.2 First, construct the feasible set or feasible region that satisfies all three constraints. The constraint for plant 1 says, "The number of doors to produce per week should not exceed 4" or "Do not use plant 1 more than 4 hours per week to produce aluminum frames" (A window does not use this plant 1): $1D+(0W) \leq 4$. Similarly, the second constraint says, "Do not use more than 12 hours per week to produce wood frames for windows each of which requires two hours to get its frame produced" (a door does not use a wood frame): $(0D)+2W \leq 12$. Finally, "Do not use more than 18 hours per week to assemble doors and windows in plant 3 when a door requires three hours to get it assembled and a windows needs 2 hours": $3D + 2W \leq 18$. Pay attention to the type of inequality; a very common stupid mistake is switching $\leq$ and $\geq$ to get funny result.

5.3 The first constraint $D \leq 4$ is the left of the vertical line of $x=4$ (the value of x is always 4 regardless of the value of y). When plugging in $(0, 0)$, you will get $0 \leq 4$ that makes sense. Therefore, choose the area that contains $(0, 0)$. The second $2W \leq 12$ is first simplified to be $W \leq 6$ when both sides of the inequality are multiplied by ½. The proper area is below the horizontal line of $f(x)=6$ (the value of $y=f(x)$ is always 6 regardless of the value of x).
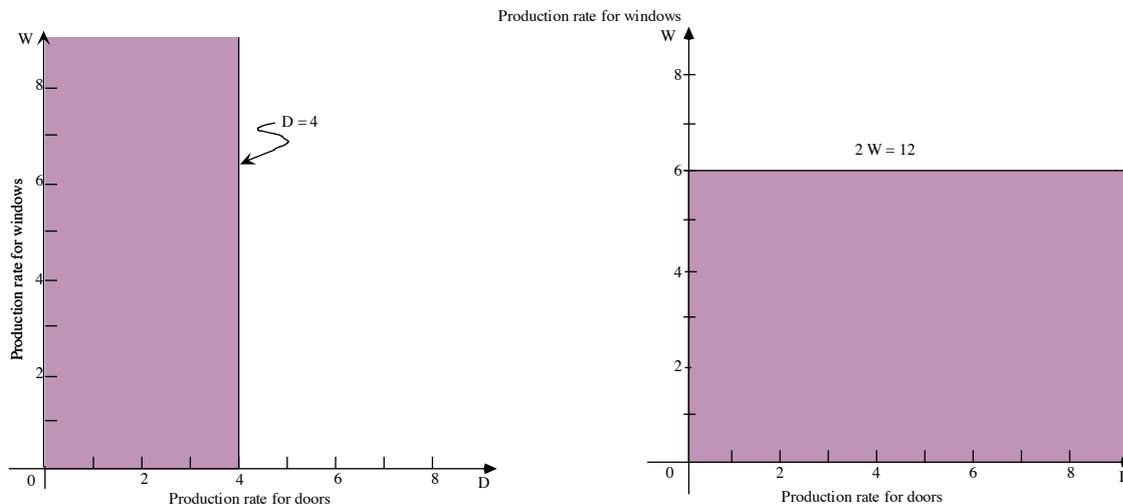
5.4 In order to determine the right area of $3D + 2W \leq 18$, rearrange the expression to have only W on the left-hand side of the inequality.

|	
$3D - 3D + 2W \leq 18 - 3D$	Cancelation law for addition
$2W \leq 18 - 3D$	
$2W \times ½ \leq (18 - 3D) \times ½$	Cancelation law for multiplication
$\mathbf{W \leq 9 - (3/2)D}$	

5.5 The function $f(x)=-(3/2)x+9$ intersects Y-axis at $(0, 9)$ and X-axis at $(6, 0)$. Since $0 \leq -(3/2)0 + 9$ and $0 \leq 9$ (when $y=W=0$ and $x=D=0$) makes sense, the right area is below the line of $f(x)=-(3/2)x+9$. Nonnegativity says both D and W should be greater than or equal

to zero. Finally, you will get the feasible region as shown in the following right figure when considering the overlapping area of all three constraints that are discussed so far.
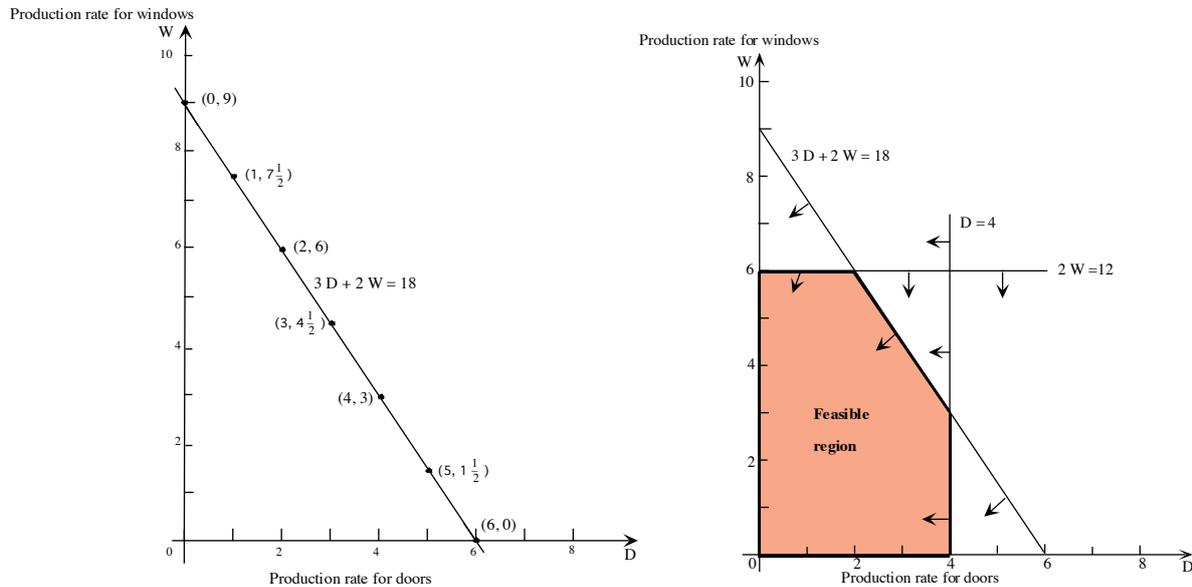
5.6 Any point in the feasible region satisfies all constraints including nonnegativity. The optimal solution exists among the points in this feasible region.

5.7 Now, consider the objective function P=300D + 500W. Rearrange the expression to have only W on the left-hand side of the equation.

P – 500W = 300D + 500W - 500W	Cancelation law for addition
P – 500W = 300D	
P – P – 500W = 300D – P	Cancelation law for addition
– 500W = 300D – P	
(– 500)×(-1/500)W = (300D – P)×(-1/500)	Cancelation law for multiplication
W = -(3/5)D + P/500	
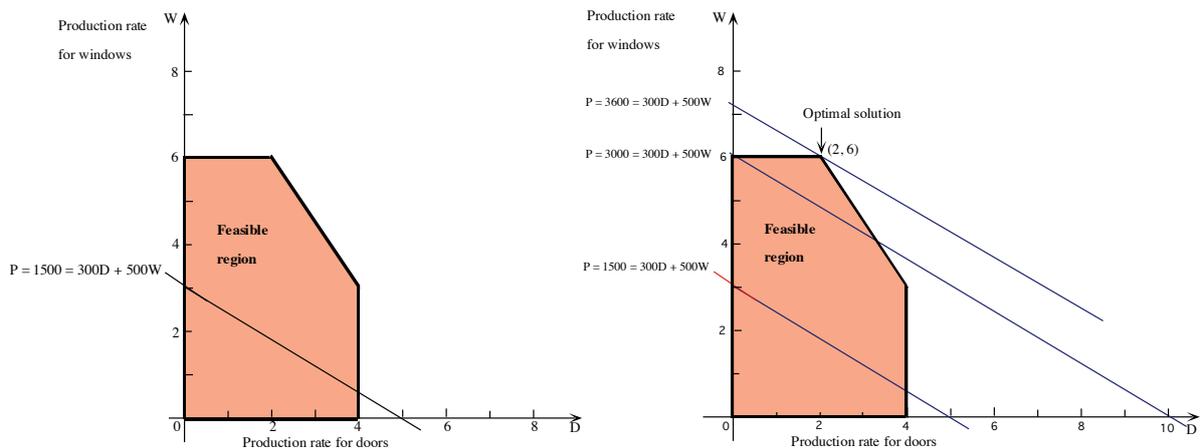
5.8 If P=3,000, this objective function line intersects the Y-axis at (0, 6) since 6=-(3/5)0+3000/500 and X-axis at (10, 0). Notice that when P increases, the objective function line will shift upward. When P decreases (e.g., P=1,500), the line will shift downward. In order to maximize the value of P, you need to <u>shift the line upward until it</u>

stays within the feasible area. The result is the point (2, 6), where the objective function line and lines of constraint 2 and 3 intersect each other. The point says that the optimal value of x (D) is 2 and 6 for y (W). At that point, P will have its maximum value of 3,600 $= 300 \times 2 + 500 \times 6$.

6. General Principles of LP Solutions

There are general rules of LP solution. A corner is an intersection of constraints and nonnegativity lines and appears on the edge of the feasible set. See the figure above. The simplex method is the underlying logic (algorithm) of LP that examines corner solutions to find the optimal solution.

6.1 The solution cannot be in the interior of the feasible set or region.

6.2 The solution must involve a corner of the feasible set (as opposed to other alternative points on the edge). The only exception is where the *iso-profit* line (objective function line) is parallel to one of the constraints (multiple solutions).

6.3 If two corners have the same best value, then all combinations between them are also solutions. This case occurs when the *iso-profit* line has the same slope as one of the constraints; the entire edge of the feasible set is optimal.

6.4 If we are searching at a corner, and the value of objective function goes down at all adjacent corners of the feasible set, you must be at the solution (this is the stopping rule that the simplex method employs).

7. Using Excel Solver for LP

7.1 Arrange data logically and systematically in an Excel worksheet. In general, begin with (1) raw data (e.g., price and coefficients), (2) decision variables, (3) the objective functions with the coefficients of decision variables, (4) constraints, and (5) negativity. If intermediate computation is needed, add to a proper place. See the following worksheet.

	A	B	C	D	E	F
1			**Wyndor Glass Co. Product-Mix Problem (H&H)**			
2						
3	Raw data		**Doors**	**Windows**		
4		Unit Profit	$300	$500		
5						
6			Hours Used per Unit Produced			Hours Available
7		Plant 1	1	0	<=	4
8		Plant 2	0	2	<=	12
9		Plant 3	3	2	<=	18
10						
11	Decision variables		x1	x2		Objective function
12			2	6		$3,600
13	Constraints					
14		Constraint 1	2		<=	4
15		Constraint 2	12		<=	12
16		Constraint 3	18		<=	18
17						
18	Nonnegativity					

7.2 You may use colors and/or border lines in order to highlight key areas.

7.3 Since spreadsheet software packages can copy even a formula in a cell, choose relative or absolute reference wisely. In the above worksheet, decision variables of doors and windows will be frequently used in the objective function and constraints. Therefore, their *absolute references* $C$12 and $D$12 is preferred to C12 and D12, respectively. $ means that its following row or column reference remains unchanged (referring to the same cell point) even when it is copied into other cells.

7.4 For instance, the objective function cell F12 may have one of the following expressions.

```
=C4*C12+D4*D12
=SUMPRODUCT(C4:D4, C12:D12)
```

7.5 The LHS of constraint 1 is provided in cell C14. The second expression below with SUMPRODUCT() is recommended. Pay attention to use of $ (you may omit $ before the column reference in this case to have C$12 and D$12).

```
=C7*$C$12+D7*$D$12
=SUMPRODUCT(C7:D7, $C$12:$D$12)
```

7.6 Then you may copy C14 into C15 and C16. In C15, for example, you will see the following expression with row references changed to 8.

```
=SUMPRODUCT(C8:D8,$C$12:$D$12)
```

7.7 Textbooks often recommend using range names, which are descriptive names assigned to a cell or range of cells in a spreadsheet. Range names seem quite useful for advanced users but not for beginners. I will minimize use of range names. Only when really needed, modeling concepts will be used as names. For instance, you may use "objective" instead of "Profit," "decision" instead of "Frames_ produced," "constraint1" instead of "Resource_used," and "RHS1" instead of "Resource1_available".

7.8 Now you are ready to run Excel Solver.[2] Click **Data→Solver** (**Tools →Solver** in Mac) to open up the Solver dialog box titled *Solver Parameters*.

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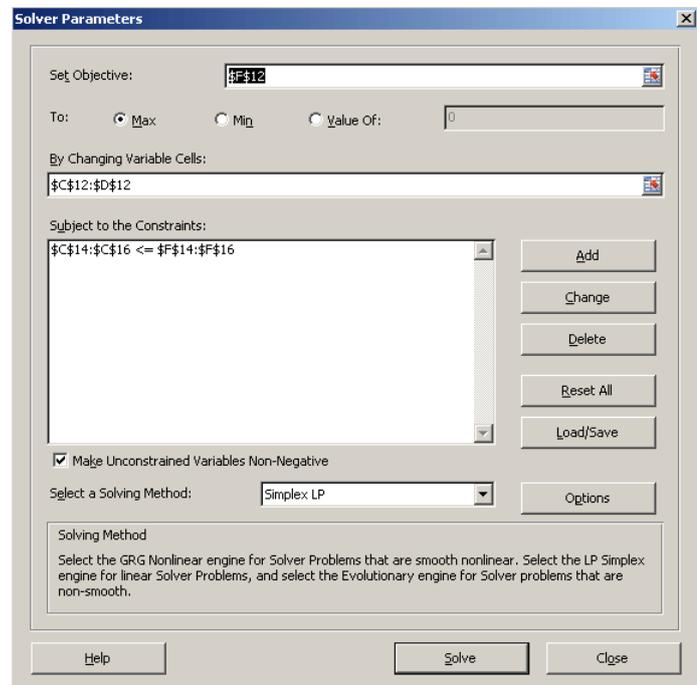
[2] If you cannot see Solver under the **Data** menu in Microsoft Excel, click consecutively **File→Options→Add-Ins→Go**, check *Solver Add-in* and then click on **OK**.

7.9 Provide the objective function cell `F12` (or `$F$12`) in *Set Objective*. Then select the type of optimization; the default type is maximization (*Max*).

7.10 Provide the range of decision variables `C12:D12` in *By Changing Variable Cells*.

7.11 Then provide the range of constraints in *Subject to the Constraints* by clicking on <u>Add</u> button on the right. In the *Add Constraint* dialog box, specify the range of LHS `C14:C16` or `$C$14:$C$16` in the *Cell Reference* input box on the left and the range of RHS `F14:F16` in *Constraint* on the right. Make sure that the inequality or equality is correctly chosen. Then click <u>OK</u> to close this dialog box; <u>Add</u> allows you to add more constraints.

7.12 Check *Make Unconstrained Variables Non-Negative* for nonnegativity. And then choose <u>Simplex LP</u> in *Select a Solving Method*. Finally, click <u>Solve</u> to optimize this LP problem.

7.13 When Excel Solver finds a solution (otherwise it will give a warning message), it will pop *Solver Results* up so that you can take the solution or not. Let us select *Keep Solver Solution* and highlight *Answer* and *Sensitivity* under *Reports* in order to generate both reports in separate worksheets. In Mac, the locations of buttons <u>OK</u> and <u>Save Scenario…</u> are switched (see the left screenshot below).
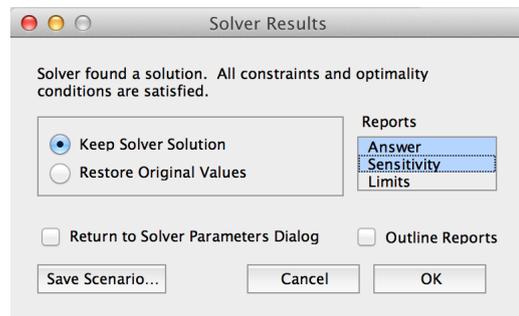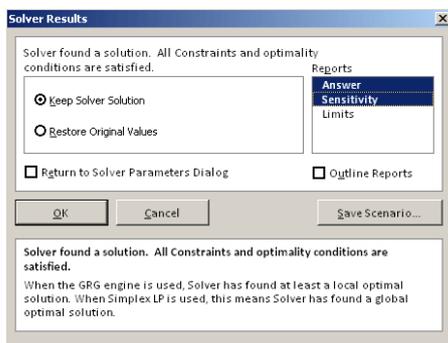
7.14 Then you will get the following worksheet of *Answer Report*.

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$F$12	Objective function	$ 3,600	$3,600

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$C$12	x1	2	2	Contin
$D$12	x2	6	6	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$C$14	Constraint 1 x1	2	$C$14<=$F$14	Not Binding	2
$C$15	Constraint 2 x1	12	$C$15<=$F$15	Binding	0
$C$16	Constraint 3 x1	18	$C$16<=$F$16	Binding	0

8. An Example of the Blending Problem

8.1 An example of 2.7 (H&H 2014) minimizes the advertisement cost while satisfying expectation (certain level of effectiveness) on the advertisement. Let

TV = the number of units of advertising on television (x_1)
PM = the number of units of advertising in the print media (x_2)

8.2 The objective function is to minimize cost of advertisement on TV and print media. Unit cost of TV advertisement is one million dollars and two million for advertisement on print media. **Minimize Cost = *TV* + 2*PM*** (in million of dollars)

8.3 There are three constraints to be satisfied. Parameters are measured in percent. Sales of the strain remover should increase by at least 3 percent (minimum percent increase in sale): ***PM ≥ 3***; sales of the liquid detergent should increase by at least 18 percent: ***3TV + 2PM ≥ 18***; sales of the powder detergent should increase by at least 4 percent: : ***−TV + 4PM ≥ 4***.

8.4 And nonnegativity says $TV \geq 0$, $PM \geq 0$. Look at the following feasible region.
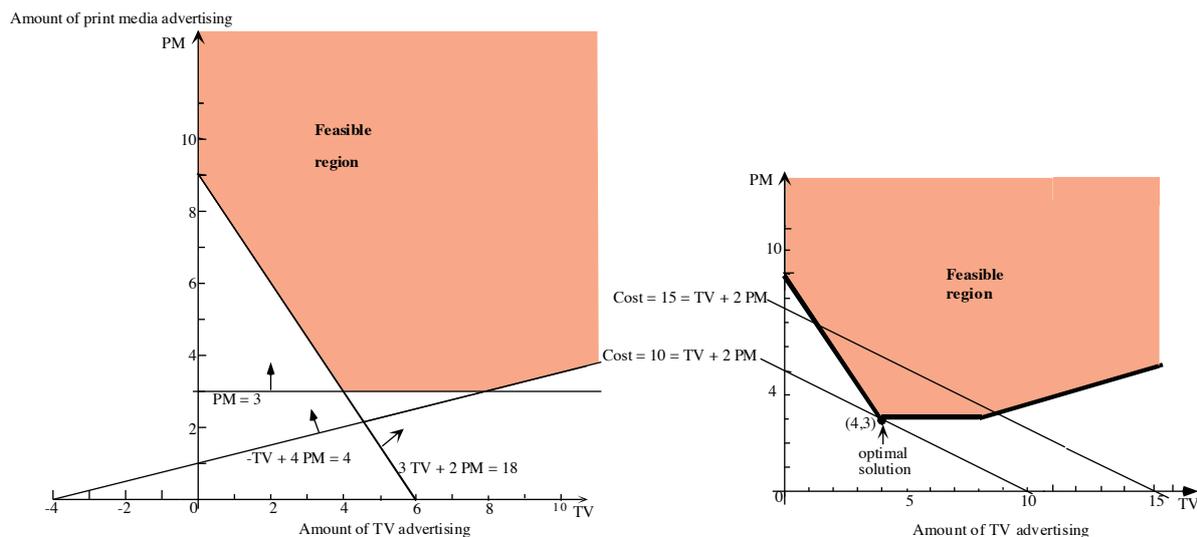
8.5 Now, check the unit of measurement in the objective function and constraints. It looks O.K.

Objective function: $ = ($/~~EA~~)×~~EA~~ + ($/~~EA~~)×~~EA~~
Constraints:
$$(\%/\text{~~EA~~})×\text{~~EA~~} \geq \%$$
$$(\%/\text{~~EA~~})×\text{~~EA~~} + (\%/\text{~~EA~~})×\text{~~EA~~} \geq \%$$
$$(\%/\text{~~EA~~})×\text{~~EA~~} + (\%/\text{~~EA~~})×\text{~~EA~~} \geq \%$$

8.6 Check the scales of parameters and decision variables. There is no too big or too small numbers. The parameters in the objective function are measured in million dollars. If they were not scaled in that way, 1,000,000 and 2,000,000 should be too big numbers. On the contrary, if they were scaled to trillion dollars, parameters might be too small numbers. The key issue is how to reasonably rescale parameters in order to avoid

overflow or underflow, which influences computation (optimization) and interpretation significantly (badly).

8.7 In order to be consistent, variable names are changed from Cost to y, TV to x_1, and PM to x_2. Therefore, LP formulation becomes as follows.

Minimize $y = x_1 + 2x_2$
subject to
$$x_2 \geq 3$$
$$3x_1 + 2x_2 \geq 18$$
$$-x_1 + 4x_2 \geq 4$$

$x_1 \geq 0, x_2 \geq 0$

8.8 When rearranging the objective function with respect to x_2, you will get $x_2 = -(1/2)x_1 + y$. The slope of iso-profit line is $-(1/2)$. When moving this iso-profit line upward and downward, you will reach at the point in the feasible regions that minimize the cost. That point is the optimal solution of $x_1=4$ and $x_2=3$.

8.9 This optimal solution is located at a corner of a feasible region in general.

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