Q.1.1

#	Green	Blue	Red	Total #
Green	950	45	5	1000
Blue	170	20	10	200
Red	0	10	40	50

Use Bayes' Theorem to find conditional probability of each state.

. $P(G|G) = \frac{950}{1000} = 0.95$

. $P(B|G) = \frac{45}{1000} = 0.045$

. $P(R|G) = \frac{5}{1000} = 0.005$

. $P(G|B) = \frac{170}{200} = 0.85$

. $P(B|B) = \frac{20}{200} = 0.10$

. $P(R|B) = \frac{10}{200} = 0.05$

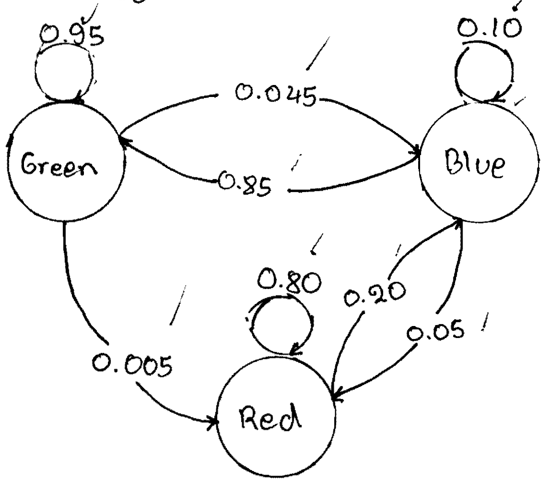
. $P(G|R) = \frac{0}{50} = 0$

. $P(B|R) = \frac{10}{50} = 0.20$

. $P(R|R) = \frac{40}{50} = 0.80$

$$P = \begin{array}{c|c} & \begin{array}{ccc} \text{Green} & \text{Blue} & \text{Red} \end{array} \\ \begin{array}{c} \text{Green} \\ \text{Blue} \\ \text{Red} \end{array} & \begin{bmatrix} 0.95 & 0.045 & 0.005 \\ 0.85 & 0.10 & 0.05 \\ 0 & 0.20 & 0.80 \end{bmatrix} \end{array}$$

Q.1.2



. Entity: Computer Operating System
. States: Green (success), Blue, Red (faile
. Interval: Hour
. Transition probability matrix:

$$P = \begin{bmatrix} 0.95 & 0.045 & 0.005 \\ 0.85 & 0.10 & 0.05 \\ 0 & 0.20 & 0.80 \end{bmatrix}$$

Q.1.3 Interpret P_{12} and P_{31} substantively

$$P = \begin{array}{c|c} & \begin{array}{ccc} \text{Green} & \text{Blue} & \text{Red} \end{array} \\ \begin{array}{c} \text{Green} \\ \text{Blue} \\ \text{Red} \end{array} & \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \end{array} = P \begin{bmatrix} 0.95 & 0.045 & 0.005 \\ 0.85 & 0.10 & 0.05 \\ 0 & 0.20 & 0.80 \end{bmatrix}$$

P_{12} : $P_{Green \to Blue}$ = 0.045 : It means that if the computer operating system is in Green (success) state this ~~hour~~ now, there is 4.5 % of chance that it will be in Blue state in next hour.

?

P_{31} = $P_{Red \to Green}$ = 0 : If the computer operating system is in Red (failure) state this ~~hour~~ now, there is 0 % (or no chance) that it will be in Green state next hour.

Q 1.4 (Excel workheet is attached)

$P^0 = [1\ 0\ 0]$, $\qquad P^5 = P^0 P^{(5)}$

$\Rightarrow t = 5$, $P^5 = [P_1^5\ \ P_2^5\ \ P_3^5] = P^4 \times P = [0.9261 \quad 0.0508 \quad 0.0230]$

If the computer operating system is in Green state this hour, there is 5.08 % of chance that it will be in Blue state 5 hours later.

Q.1.5 Interpret the first column of $P^{(20)}$ substantively.

$$P^{(20)} = \begin{array}{c} \\ \text{Green} \\ \text{Blue} \\ \text{Red} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Green} & \text{Blue} & \text{Red} \end{array} \\ \begin{bmatrix} 0.9109 & 0.0534 & 0.0356 \\ 0.9099 & 0.0536 & 0.0364 \\ 0.8956 & 0.0560 & 0.0483 \end{bmatrix} \end{array}$$

$P_{11}^{(20)}$ = If the computer operating system is in Green state in a particular hour, there is 91.09 % of chance that it will remain in Green state in 20 hours later.

$P_{21}^{(20)}$ = If the computer operating system is in Blue state in particular hour, there is 90.99 % of chance that it will be in Green state in 20 hours later.

$P_{31}^{(20)}$ = If the computer operating system is in Red state in particular hour, there is 89.56 % of chance that it will be in Green state in 20 hours later.

Q.1.6

$$\Rightarrow \Pi = \Pi P \Rightarrow [\Pi_1\ \ \Pi_2\ \ \Pi_3] = [\Pi_1\ \ \Pi_2\ \ \Pi_3] \times \begin{bmatrix} 0.95 & 0.045 & 0.005 \\ 0.85 & 0.10 & 0.05 \\ 0 & 0.20 & 0.80 \end{bmatrix}$$

we can draw the following four equation from the above.

$\pi_1 = 0.95\pi_1 + 0.5\pi_2 + 0.\pi_3$ (1)

$\pi_2 = 0.045\pi_1 + 0.10\pi_2 + 0.20\pi_3$ (2)

$\pi_3 = 0.005\pi_1 + 0.05\pi_2 + 0.80\pi_3$ (3)

$\pi_1 + \pi_2 + \pi_3 = 1$ (4)

Then, let's take two equations (2 out of 3) and $\pi_1 + \pi_2 + \pi_3 = 1$ and rearrange it into $BX = Y$ form.

(2) $\pi_2 = 0.045\pi_1 + 0.10\pi_2 + 0.20\pi_3$ (Cancellation law by adding $-\pi_2$)

$\pi_2 - \pi_2 = 0.045\pi_1 + 0.10\pi_2 - \pi_2 + 0.20\pi_2$

$0 = 0.045\pi_1 - 0.9\pi_2 + 0.20\pi_3$

or $0.045\pi_1 - 0.9\pi_2 + 0.20\pi_3 = 0$

(3) $\pi_3 = 0.005\pi_1 + 0.05\pi_2 + 0.80\pi_3$ (cancellation low by adding $-\pi_3$)

$\pi_3 - \pi_3 = 0.005\pi_1 + 0.05\pi_2 + 0.80\pi_3 - \pi_3$

$0 = 0.005\pi_1 + 0.05\pi_2 - 0.2\pi_3$

or $0.005\pi_1 + 0.05\pi_2 - 0.2\pi_3 = 0$

(4) $\pi_1 + \pi_2 + \pi_3 = 1$

$$\Rightarrow BX = Y = \begin{bmatrix} 0.045 & -0.9 & 0.20 \\ 0.005 & 0.05 & -0.20 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

0.1.7 Steady-state probabilities (see excel workseheet for full calculation)

$X(\pi): \pi_1 \ \pi_2 \ \pi_3 = [0.9103 \quad 0.0535 \quad 0.0361]$

0.1.8 Interpret steady-state probabilities.

On average, there is 91.03% of chance that the operating system is in Green state, 5.35% of chance is in Blue state and 3.61% of chance in Red state in the long run.

In my opinion, this operating systeme is suck (not good enough), because the failure rate is higher than the claim rate (3.61% > 0.1%)

Q.2.1 Construct the transition probability matrix.

$$P = \begin{array}{c|ccccc} & \text{Normal} & \text{Overdue} & \text{Delinguent} & \text{Fully Paid} & \text{Default} \\ \hline \text{Normal} & 0.2 & 0.3 & 0 & 0.5 & 0 \\ \text{Overdue} & 0 & 0.3 & 0.3 & 0.4 & 0 \\ \text{Delinguent} & 0 & 0 & 0.4 & 0.1 & 0.5 \\ \text{Fully Paid} & 0 & 0 & 0 & 1 & 0 \\ \text{Default} & 0 & 0 & 0 & 0 & 1 \end{array}$$

Q.2.2 Report Q and R.

$$Q = \begin{bmatrix} 0.2 & 0.3 & 0 \\ 0 & 0.3 & 0.3 \\ 0 & 0 & 0.4 \end{bmatrix} \qquad R = \begin{bmatrix} 0.5 & 0 \\ 0.4 & 0 \\ 0.1 & 0.5 \end{bmatrix}$$

Q 2.3 (excel worksheet is attached)

$$(I-Q)^{-1} = \begin{array}{c} \\ N \\ O \\ D \end{array} \begin{array}{c} \begin{array}{ccc} N & O & D \end{array} \\ \begin{bmatrix} 1.25 & 0.5357 & 0.2678 \\ 0 & 1.4285 & 0.7142 \\ 0 & 0 & 1.6666 \end{bmatrix} \end{array}$$

Fundamental matrix

Q.2.4 Explain the first row of this fundamental matrix substantively

The first row of this fundamental matrix $(I-Q)^{-1}$ says that on average a normal taxpayer will spend 1.25 quarters in the normal state, 0.5357 quarter in overdue and 0.2678 quarter in delinguent before being absorbed in fully paid or default eventually. In order word, a normal taxpayer will spend on average 2.0535 (=1.25+0.5357+0.2678) quarters before being fully paid or default.

Q.2.5 (excel worksheet is attached)

$$(I-Q)^{-1}R = \begin{array}{c} \\ N \\ O \\ D \end{array} \begin{array}{c} \begin{array}{cc} F & De \end{array} \\ \begin{bmatrix} 0.8660 & 0.1339 \\ 0.6428 & 0.3571 \\ 0.1666 & 0.8333 \end{bmatrix} \end{array}$$

Absorbing probabilities matrix

Q.2.6 Explain the second column of this absorbing probabilities matrix.

The probabilities that a normal, overdue and delinguent taxpayer will be default eventually, are 13.39 %, 35.71 % and 83.33 %, respectively.

Q.2.7 Suppose the amount of tax imposed in:

- Normal is on average $ 500 M
- Overdue ——— " ——— $ 200 M
- Delinquent ——— " ——— $ 50 M

\# The amount of final tax collection is estimated to be $ 570 M that is decomposed as.

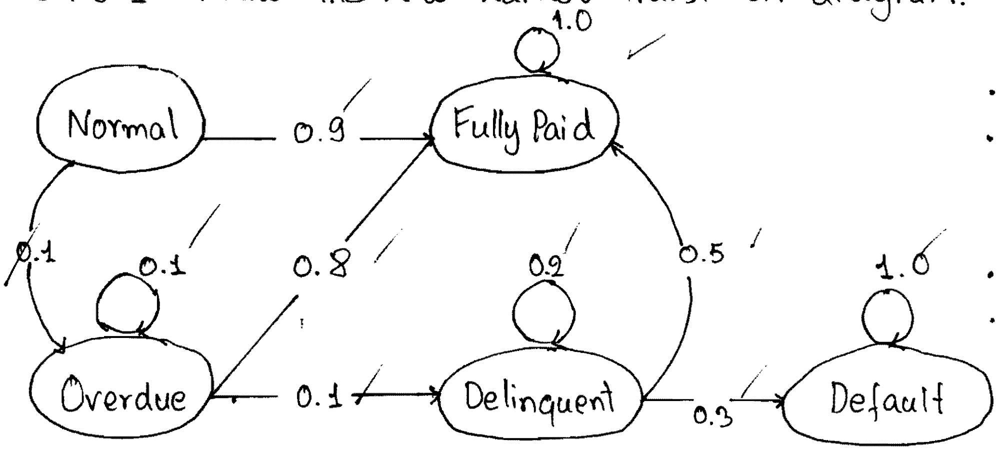
- $433M = $500M x 0.8660 (from those who are normal)
- $ 128.56M = $ 200M x 0.6428 (from those who are overdue)
- $ 8.33 M = $ 50 M x 0.1666 (from those who are delinquent)

$569.89 M (= $ 433 M + $128.56M + $8.33 M

\# The amount of default is estimated to be $180M = $66.95M + $71.42M + $41.665 M

- $ 66.95 M = $500 M x 0.1339 (from those who are normal)
- $ 71.42 M = $200 M x 0.3571 (from those who are overdue)
- $ 41.665 M = $ 50 M x 0.8333 (from those who are delinquent)

Q.3.1 Draw the new Markov transition diagram.



- Entity: Tax collection system
- States: Normal, Overdue, Delinquent, Fully Paid and Default.
- Inteval: Quarter (three mont
- Transition probability matrix:

$$P = \begin{array}{c|ccccc} & N & O & D & F & De \\ \hline N & 0 & 0.1 & 0 & 0.9 & 0 \\ O & 0 & 0.1 & 0.1 & 0.8 & 0 \\ D & 0 & 0 & 0.2 & 0.5 & 0.3 \\ F & 0 & 0 & 0 & 1 & 0 \\ De & 0 & 0 & 0 & 0 & 1 \end{array}$$

Q.3.2 Report new Q and R

$$Q = \begin{bmatrix} 0 & 0.1 & 0 \\ 0 & 0.1 & 0.1 \\ 0 & 0 & 0.2 \end{bmatrix}, \quad R = \begin{bmatrix} 0.9 & 0 \\ 0.8 & 0 \\ 0.5 & 0.3 \end{bmatrix}$$

Q 3.3 (Excel worksheet is attached)

$$(I-Q)^{-1} = \begin{bmatrix} 1 & 0.1111 & 0.0138 \\ 0 & 1.1111 & 0.1388 \\ 0 & 0 & 1.25 \end{bmatrix} , \quad (I-Q)^{-1}R = \begin{bmatrix} 0.9958 & 0.0041 \\ 0.9583 & 0.0416 \\ 0.625 & 0.375 \end{bmatrix}$$

Q.3.4 According to question 2.7, the amount of tax impose in N = \$500 M, O = \$200 M and D = \$50 M, respectively.

\# Find the new amount of tax collection after "EZ tax" is estimated to be \$720M

- \$ 497.9 M = 500 × 0.9958 (from those who are Normal)
- \$ 191.66 M = 200 × 0.9583 (from those who are overdue)
- \$ 31.25 M = 50 × 0.625 (from those who are Delinquent)

\$ 720.81 M = 497.9 + 191.66 + 31.25

\# Find the new amount of default after "EZ tax" is estimated to be \$ 29. M

- \$ 2.05 M = 500 × 0.0041 (from those who are Normal)
- \$ 8.32 M = 200 × 0.0416 (from those who are Overdue)
- \$ 18.75 M = 50 × 0.375 (from those who are Delinquent).

\$ 29.12 M = 2.05 + 8.32 + 18.75

Q.3.5 Base on question 2.7 and 3.4, we can compute:

- Expected net tax revenue before "EZ tax" = \$570M (total tax payment)
- Expected ____ " ____ after "EZ tax" = \$720 M (total payment) − \$10M (total cost)
  = \$ 710 M

The impact of new policy "EZ tax" is = 710 − 570 = \$ 140 M

The positive value indicates that new policy was obviouslly successful. As a new director's expectation, new policy drastically change the transition probability and its impact is larger as well. The tax revenue increase by \$ 140 M, therefore we would support the "EZ tax" to increase revenue (tax payment).