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Linear Programming Duality Theory

"[E]very linear programming problem has associated with it another linear programming problem called the dual" (p. 197). Duality theorem says that the value of the objective functions of *primal problem* and *dual problem* is identical.

1. Primal Problem versus Dual Problem

"[T]he dual problem uses exactly the same *parameters* as the primal problem, but in different locations…" (p. 198). "[W]ith the primal problem in *maximization* form, the dual problem is in *minimization* form instead" (p. 198). Primal (decision) variables x_i and dual (decision) variables y_j are switched; objectives and constraints are exchanged.

- "The coefficients in the objective function of the primal problem are the *right-hand sides* of the functional constraints in the dual problem" (p. 198).
- "The right-hand sides of the functional constraints in the primal problem are the coefficients in the objective function of the dual problem" (p. 198).
- "The coefficients of a variable in the functional constraints of the primal problem are the coefficients in a functional constraint of the dual problem" (p. 198).

	Primal Problem	Dual Problem
Objective Function	Max $Z = \sum_{j=1}^{n} c_j x_j$	Min $W = \sum_{i=1}^{m} b_i y_i$
Constraints (Subject to)	$\sum_{j=1}^{n} a_{ij} x_j \le b_i$	$\sum_{i=1}^{m} a_{ij} y_i \ge c_j$
Nonnegativity	$x_j \ge 0$, i $(1\ldots m)$, j $(1\ldots n)$	$y_j \ge 0$, i $(1\ldots m)$, j $(1\ldots n)$

Example

	Primal Problem	Dual Problem
Objective Function	Max $Z = 3x_1 + 5x_2$	Min $W = 4y_1 + 12y_2 + 18y_3$
Constraints (Subject to)	$x_1 \le 4$	$y_1 + 3y_3 \ge 3$
	$2x_2 \le 12$	$2y_2 + 2y_3 \ge 5$
	$3x_1 + 2x_2 \le 18$	
Nonnegativity	$x_j \ge 0$, i $(m=3)$, j $(n=2)$	$y_j \ge 0$, i $(m=3)$, j $(n=2)$

2. Duality theorem

- If one problem has *feasible solutions* and a *bounded* objective function (and so has an optimal solution), then so does the other problem, so both the weak and strong duality properties are applicable.

- If one problem has *feasible solutions* and an *unbounded* objective function (and so *no optimal solution*), then the other problem has *no feasible solutions*.
- If one problem has *no feasible solutions*, then the other problem has either *no feasible solutions* or an *unbounded* objective function.

Some important properties are:

- "**Weak duality property**: If *x* is a feasible solution for the primal problem and *y* is the feasible solution for the dual problem" (p. 203).
- **"Strong duality property**: If x^* is an optimal solution for the primal problem and y^* is an optimal solution for the dual problem" (p. 203).
- "**Complementary basic solutions property**: Each *basic* solution in the *primal problem* has a **complementary basic solution** in the *dual problem*, where their respective objective function values (Z and W) are equal" (p. 209).
- "**Complementary optimal basic solutions property**: An optimal solution in the primal problem has a **complementary optimal basic solution** in the *dual problem*, where their respective objective function values (Z and W) are equal" (p. 211).

3. Interpretation of Duality

- "The dual variable y_j is interpreted as the contribution to profit per unit of source *i* (*i* = 1, 2, … *m*), when the current set of basic variables is used to obtain the primal solution. In other words, the y_j values (or y^*_j values in the optimal solution) are just the **shadow prices**" (p. 206).
- "$\Sigma a_{ij}y_j$ is interpreted as the current contribution to profit of the mix of resources that would be consumed if 1 unit of activity j were used" (p. 206).
- "$\Sigma a_{ij}y_j \geq c_j$ says that the actual contribution to profit of the above mix of resources must be at least as much as if they were used by 1 unit of activity j; otherwise, we would not be making the best possible use of these resources" (p. 206).

References

Hillier, Frederick S., and Lieberman, Gerald J. 2015. *Introduction to Operations Research*, 10th ed. Singapore: McGraw-Hill Education.

<u>*End of the final exam*</u>.